UNIVERSITY OF TORONTO
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ECO 2030, unit 1 (Microeconomic Theory II)
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Duration: 2 hour 50 minutes

No aids allowed

This examination paper consists of **8** pages and **7** questions. Please bring any discrepancy to the attention of an invigilator. The number in brackets at the start of each question is the number of points the question is worth.

Answer all questions.

TO OBTAIN CREDIT, YOU MUST GIVE ARGUMENTS TO SUPPORT YOUR ANSWERS.

1. [15] For every value of a other than $a = 1$ and $a = 2$, find all the Nash equilibria in pure and mixed strategies of the following strategic game.

	L	C	R
T	$1, 3$	$0, 4$	$2, 2$
M	$0, 2$	$1, 0$	$4, a$
B	$3, 0$	$0, 2$	$1, a$

Solution: First note that T is strictly dominated by the mixed strategy that assigns probability $\frac{1}{2}$ to M and probability $\frac{1}{2}$ to B. So in every Nash equilibrium of the game, player 1 assigns probability 0 to T.

Subsequently, restrict attention to the game in which player 1 has only the actions M and B:

	L	C	R
M	$0, 2$	$1, 0$	$4, a$
B	$3, 0$	$0, 2$	$1, a$

- If $a > 2$ then R strictly dominates L and C, so the game has a unique Nash equilibrium, (M, R).
- If $a < 1$ then R is strictly dominated by the mixed strategy that assigns probability $\frac{1}{2}$ to L and probability $\frac{1}{2}$ to C. The game

	L	C
M	$0, 2$	$1, 0$
B	$3, 0$	$0, 2$

  has a unique Nash equilibrium, $((\frac{1}{2}, \frac{1}{2}), (\frac{1}{4}, \frac{3}{4}))$, so that the original game has a unique Nash equilibrium, $((0, \frac{1}{2}, \frac{1}{2}), (\frac{1}{4}, \frac{3}{4}, 0))$.
- If $1 < a < 2$ then the game has no pure strategy equilibrium. Consider mixed strategy equilibria. Denote by p the probability player 1 assigns to M.

It has no equilibrium in which the support of player 2's strategy is $\{L, C\}$ because then we need $p = \frac{1}{2}$, in which case player 2's payoff to R exceeds her payoff to L and C.
For an equilibrium in which the support of player 2's strategy is $\{L, R\}$ we need $p = a/2$, in which case her payoffs to L, C, and R are a, $2 - a$, and a; $2 - a \leq a$ given $a \geq 1$. For player 1 to get the same expected payoff for each of her actions, we need player 2 to choose L and R each with probability $\frac{1}{2}$. Thus the original game has a Nash equilibrium $((0, a/2, 1 - a/2), (\frac{1}{2}, 0, \frac{1}{2}))$.
If the support of player 2's strategy is $\{C, R\}$, then player 1's payoff to M exceeds her payoff to B, so that the game has no Nash equilibrium of this type.
For no strategy of player 1 are player 2's payoffs to L, C, and R all equal (given $a > 1$), so the game has no Nash equilibrium of this type.

2. A pie of size 1 is available. Simultaneously, each of n players, 1, ..., n, requests an amount of the pie (a number in $[0, 1]$), and the pie is divided as follows. First, the player whose request is smallest is assigned the amount she requested. Then, any remaining pie is assigned to the player with the next smallest request, up to the value of her request. The assignment process continues in the same manner as long as pie remains. Any ties are broken in favor of the player with the smallest index. (For example, if $n = 4$ and the players' requests are $(\frac{1}{4}, \frac{1}{3}, \frac{1}{4}, \frac{2}{3})$, then player 1 is assigned $\frac{1}{4}$, player 3 is assigned $\frac{1}{4}$, player 2 is assigned $\frac{1}{3}$, and player 4 is assigned $\frac{1}{6}$. If the requests are $(1, \frac{2}{3}, \frac{2}{3}, \frac{3}{4})$, then player 2 is assigned $\frac{2}{3}$, player 3 is assigned $\frac{1}{3}$, and players 1 and 4 are assigned 0.)

   (a) [7] Find the set of actions, if any, that are strictly dominated by another action, and the set of actions that are weakly dominated by another action. (If you claim an action is dominated, you need to show why it is dominated. If you claim an action is not dominated, you need to show why it is not dominated.)

   Solution: Any action less than $1/n$ is strictly dominated by the action $1/n$. For any actions of the other players, the action $1/n$ yields the payoff $1/n$ whereas any action less than $1/n$ yields a payoff less than $1/n$.
   No action of at least $1/n$ is weakly or strictly dominated by another action. Suppose a player requests $1/n$. Then her payoff is $1/n$ regardless of the other players' actions. If she requests $x < 1/n$, her payoff is x regardless of the other players' actions. Suppose she requests $1/n + \varepsilon$, with $\varepsilon > 0$. Then if every other player requests $1/n + \varepsilon/2$, her payoff is less than $1/n$. So requesting $1/n$ is not weakly dominated by any other action.
   Now suppose that a player requests $x > 1/n$. Then if every other player requests x, the player's payoff is x, whereas if the player requests less than x then her payoff is less than x. So requesting x is not weakly or strictly dominated by requesting less than x. It is also not weakly or strictly dominated by requesting any amount $y > x$, because for some requests of the other players between x and y, requesting y yields a lower payoff than requesting x.

(b) [8] Find the pure strategy Nash equilibria of the game that models the situation.

Solution: By part (a), in any Nash equilibrium every player's request is at least $1/n$. The game has no Nash equilibrium in which the requests are not all the same, because then a player making the smallest request can increase her request slightly and improve her payoff. If the common request is greater than $1/n$ then player n receives less than $1/n$ and can increase her payoff by reducing her request slightly.

Thus in any Nash equilibrium, every player requests $1/n$. This action profile is a Nash equilibrium because a player's reducing her request reduces her payoff and a player's increasing her request has no effect on her payoff.

3. Two individuals choose how much effort to expend. Effort is a nonnegative number, and player 1's payoff function is $e_1(1+e_2-e_1)$, where e_1 is player 1's effort level and e_2 is player 2's effort level. Player 2's cost of effort is either low, in which case her payoff function is $e_2(1+e_1-e_2)$, or high, in which case her payoff function is $e_2(1+e_1-2e_2)$. Player 2 knows her cost of effort, but player 1 does not know player 2's cost of effort. Player 1 believes that player 2's cost of effort is low with probability p and high with probability $1-p$, where $0 < p < 1$.

(a) [5] Model this situation as a Bayesian game.

Solution: Here is a model of the situation as a Bayesian game:

Players $N = \{1, 2\}$.

States $\{L, H\}$ (player 2's effort cost)

Actions $A_1 = A_2 = \mathbb{R}_+$.

Signals $T_1 = \{z\}$, $T_2 = \{l, h\}$. $\tau_1(s) = z$, $\tau_2(s) = s$.

Beliefs Each player's prior is that the state is L with probability p.

Payoffs The payoff of player 1 is $e_1(1+e_2-e_1)$ and the payoff function of player 2 is $e_2(1+e_1-e_2)$ in state L and $e_2(1+e_1-2e_2)$ in state H.

(b) [10] Find the Nash equilibria of the game as a function of p.

Solution: First find player 1's best response function. Her best response to player 2's strategy is the solution of

$$\max_{e_1 \geq 0} pe_1(1+e_2(L)-e_1) + (1-p)e_1(1+e_2(H)-e_1)$$

or

$$\max_{e_1 \geq 0} e_1(1-e_1+pe_2(L)+(1-p)e_2(H)).$$

So player 1's best response function is given by

$$b_1(e_2(L), e_2(H)) = \tfrac{1}{2}(1+pe_2(L)+(1-p)e_2(H)).$$

Now find the best response function of each type of player 2. The low type's problem is

$$\max_{e_2 \geq 0} e_2(1+e_1-e_2),$$

so her best response function is given by

$$b_2^L(e_1) = \tfrac{1}{2}(1 + e_1).$$

Similarly, the best response function of the high type of player 2 is given by

$$b_2^H(e_1) = \tfrac{1}{4}(1 + e_1).$$

The Nash equilibria are given by the solutions of the equations

$$\begin{aligned} e_1 &= b_1(e_2(L), e_2(H)) \\ e_2(L) &= b_2^L(e_1) \\ e_2(H) &= b_2^H(e_1). \end{aligned}$$

These equations have a unique solution,

$$\begin{aligned} e_1 &= \frac{5+p}{7-p} \\ e_2(L) &= \frac{6}{7-p} \\ e_2(H) &= \frac{3}{7-p}. \end{aligned}$$

4. Player 1 has one unit of a good. She chooses how much to give player 2. Player 2 transforms any amount x into kx, where $k > 1$, and then chooses how much of this larger amount to give back to player 1. If player 1 initially gives x to player 2 and player 2 returns y to player 1, then the payoff of player 1 is $1 - x + y$ and the payoff of player 2 is $kx - y$.

   (a) [4] Find the subgame perfect equilibria of the extensive game that models this situation. (Be sure to give the equilibrium *strategies*.)

   Solution: In the subgame following player 1's choice of x, player 2's optimal action is $y = 0$ for every value of x. Given this strategy of player 2, player 1's optimal action is $x = 0$. Thus the game has a unique subgame perfect equilibrium, in which player 1's strategy is $x = 0$ and player 2's strategy selects $y = 0$ after every history x.

   (b) [4] Does the game have any Nash equilibrium in which the outcome differs from the outcome of any subgame perfect equilibrium?

   Solution: No, because if player 1 gives a positive amount, say x, to player 2, then player 2 optimally gives nothing back to player 1, and player 1 is thus better off giving nothing.

   Now suppose that player 2's payoff is not $kx - y$, but $y(kx - y)$.

   (c) [7] The nature of the subgame perfect equilibria under this assumption depend on the value of k. For each possible value of k, give the equilibrium *strategies* as well as the equilibrium outcome.

Solution: In the subgame following player 1's choice of x, player 2's payoff is $y(kx - y)$. Thus player 2's optimal action is $y = \frac{1}{2}kx$.

Now consider player 1's choice at the start of the game. Her payoff is $1 - x + y = 1 - x + \frac{1}{2}kx = 1 + (\frac{1}{2}k - 1)x$. Thus if $k > 2$ then her optimal action is $x = 1$, if $k = 2$ then every value of x is optimal, and if $k < 2$ then her optimal action is $x = 0$.

Thus in every subgame perfect equilibrium player 2's strategy chooses $\frac{1}{2}kx$ in the subgame following player 1's choice of x.

If $k > 2$ then player 1 chooses $x = 1$, if $k < 2$ then she chooses $x = 0$, and if $k = 2$ then for every z with $0 \leq z \leq 1$ the game has a subgame perfect equilibrium in which player 1 chooses $x = z$.

So for every z with $0 \leq z \leq 1$ the following strategy pair is a subgame perfect equilibrium of the game:

Player 1's strategy

$$\begin{cases} 0 & \text{if } k < 2 \\ z & \text{if } k = 2 \\ 1 & \text{if } k > 2. \end{cases}$$

Player 2's strategy $\frac{1}{2}kx$ for each value of x.

If $k > 2$ the outcome of the subgame perfect equilibrium is that $x = 1$ and $y = \frac{1}{2}k$, if $k < 2$ the outcome is that $x = 0$ and $y = 0$, and if $k = 2$ then in every subgame perfect equilibrium player 1 chooses some value of x and player 2 chooses $y = \frac{1}{2}kx$.

5. Consider a variant of the Bargaining Game of Alternating Offers in which each player cares not only about her monetary payoff but also about the monetary payoff of the other player. Specifically, assume that the payoff of each player i for the division of the pie (x_1, x_2) (where $x_1 + x_2 = 1$) received in period t is $\delta^t(x_i + \lambda x_j)$, where j is the other player, $0 < \delta < 1$, and $0 < \lambda < 1$. (The first period is period 0. Both players have the same discount factor, δ.)

   (a) [12] Find a subgame perfect equilibrium of this game. (Be sure to specify the equilibrium **strategies** fully. You may take as given the fact that a strategy pair is a subgame perfect equilibrium if and only if it satisfies the one-deviation property. You need to show that the strategy pair you find is a subgame perfect equilibrium.)

   Solution: As in the standard model, look for a stationary equilibrium in which each player is indifferent between accepting and rejecting a proposal. For given proposals x^* and y^*, let s_1^* be the strategy of player 1 that always proposes x^* and accepts a proposal y if and only if $y_1 \geq \delta x_1^*$, and let s_2^* be the strategy of player 2 that always proposes y^* and accepts a proposal x if and only if $x_2 \geq \delta y_x^*$. Now look for proposals x^* and y^* such that each player is indifferent between accepting a proposal and rejecting it. In such an equilibrium we need

$$x_2^* + \lambda x_1^* = \delta(y_2^* + \lambda y_1^*)$$
$$y_1^* + \lambda y_2^* = \delta(x_1^* + \lambda x_2^*)$$

which yields

$$x_1^* = \frac{1 - \delta\lambda}{(1-\lambda)(1+\delta)}$$
$$y_1^* = \frac{\delta - \lambda}{(1-\lambda)(1+\delta)}.$$

I argue that the strategy pair (s_1^*, s_2^*) for which x^* and y^* take these values is a subgame perfect equilibrium. The argument is the same as the standard one.

- If player 1 increases the amount she offers player 2, player 2 accepts the new offer and player 1 is worse off.
- If player 1 decreases the amount she offers player 2, player 2 rejects the new offer and player 1 gets $y_1^* + \lambda y_2^*$ with one period of delay, instead of $x_1^* + \lambda x_2^*$, and thus if worse off.
- If player 1 is faced with a proposal y in which $y_1 > y_1^*$, her strategy calls for her to accept the offer, in which case she gets $y_1 + \lambda y_2$. If instead she rejects the offer, she gets $x_1^* + \lambda x_2^*$ with one period of delay, which is worth $y_1^* + \lambda y_2^* = \lambda + (1-\lambda)y_1^* < \lambda + (1-\lambda)y_1 = y_1 + \lambda y_2$, so she is worse off than she is when she accepts the offer.
- If player 1 is faced with a proposal y in which $y_1 < y_1^*$, her strategy calls for her to reject the offer, in which case she $x_1^* + \lambda x_2^*$ with one period of delay, which is worth $y_1^* + \lambda y_2^*$. If instead she accepts the offer, she gets $y_1 + \lambda y_2 = \lambda + (1-\lambda)y_1 < \lambda + (1-\lambda)y_1^* = y_1^* + \lambda y_2^*$, so she is worse off than she is when she rejects the offer.

A symmetric argument applies to player 2's strategy.

(b) [3] Does the equilibrium allocation become more or less unequal at λ increases? Why does the model produce this result?

Solution: As λ increases, x_1^* increases. That is, the equilibrium allocation becomes less equal. The reason is that as player 2 cares more about player 1's payoff, player 1 has to give less of the good to player 2 for player 2 to accept a proposal.

6. (a) [5] Consider the infinitely repeated *Prisoner's Dilemma* in which each player's discount factor is δ, with $0 < \delta < 1$, and the stage game payoffs are given in the following figure.

	C	D
C	$2, 2$	$0, 3$
D	$3, 0$	$1, 1$

Consider the following strategy s. Choose C in period 1 and after any history in which the other player chose C in every period except, possibly, the previous period; choose D after any other history. (The initiation of punishment is delayed by one period.)

Determine the values of δ, if any, for which the strategy pair (s, s) is a Nash equilibrium of the infinitely repeated game.

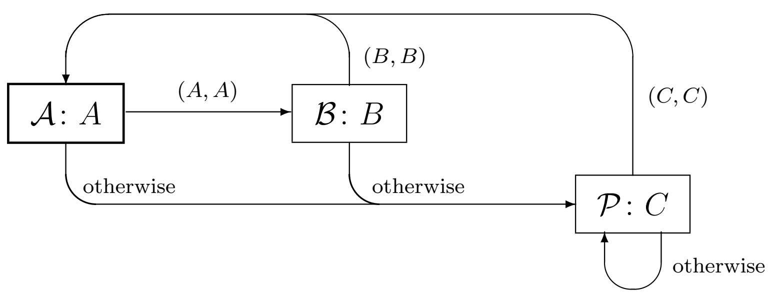
Solution: A player who adheres to the strategy obtains the discounted average payoff of 2. The best deviation yields the stream of payoffs $(3, 3, 1, 1, \ldots)$, with a discounted average of $3(1-\delta)(1+\delta)+\delta^2$. Thus for an equilibrium we require $3(1-\delta)(1+\delta)+\delta^2 \leq 2$, or $\delta \geq \frac{1}{2}\sqrt{2}$.

(b) [10] Consider the infinitely repeated game with discounting of the following strategic game.

	A	B	C
A	$4,4$	$3,0$	$1,0$
B	$0,3$	$2,2$	$1,0$
C	$0,1$	$0,1$	$0,0$

When the discount factor is close to 1, does this game have a subgame perfect equilibrium in which the outcome is (A, A) in every odd period and (B, B) in every even period? Either show that it has no such subgame perfect equilibrium or specify a strategy pair that is a subgame perfect equilibrium and find conditions on the discount factor for which the strategy pair is a subgame perfect equilibrium.

Solution: Consider the following strategy.



Consider the conditions under which the strategy pair in which each player uses this strategy is a subgame perfect equilibrium. Both players' strategies are always in the same state, so consider deviations in each state.

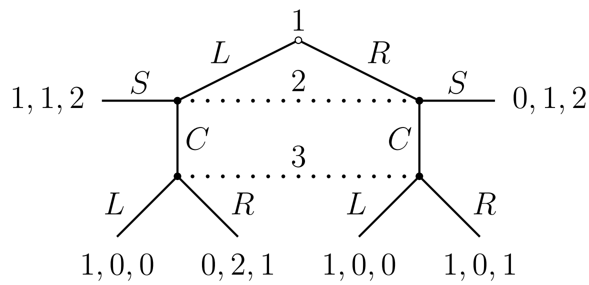
State $\mathcal{A}$ If a player adheres to the strategy, her stream of payoffs is $4, 2, 4, 2, \ldots$. If she deviates once and then returns to her strategy, her stream of payoffs is $0, 0, 0, 4, 2, 4, 2, \ldots$, so she is worse off deviating.

State $\mathcal{B}$ If a player adheres to the strategy, her stream of payoffs is $2, 4, 2, 4, \ldots$. If she deviates once and then returns to her strategy, her best stream of payoffs is $3, 0, 4, 2, 4, 2, \ldots$, so she is no better off deviating if $-1+4\delta+2\delta^2(-1+\delta-\delta^2+\delta^3-\ldots) \geq 0$, or $\delta \geq \frac{1}{4}(\sqrt{17}-3) > 0$.

State $\mathcal{P}$ If a player adheres to the strategy, her stream of payoffs is $0, 4, 2, 4, 2, \ldots$. If she deviates once and then returns to her strategy, her best stream of payoffs is $1, 0, 4, 2, 4, 2, \ldots$, so she is no better off deviating if $-1 + 4\delta + 2\delta^2(-1 + \delta - \delta^2 + \delta^3 - \ldots) \geq 0$, the same condition as for state $\mathcal{B}$.

Thus the strategy pair is a subgame perfect equilibrium if δ is close enough to 1.

7. Consider the following extensive game.



(a) [5] Find a pure strategy weak sequential equilibrium of this game. (Remember to specify both the strategy profile and the beliefs.)

Solution: In any WSE, player 3 chooses R, because regardless of her belief doing so gives her a payoff of 1, whereas choosing L gives her a payoff of 0.
So player 2 chooses C if the probability she assigns to the history L is greater than $\frac{1}{2}$, S if this probability is less than $\frac{1}{2}$, and either C or S if the probability is $\frac{1}{2}$.
If player 2 chooses C, then player 1's optimal action is R, in which case the consistency of player 2's belief requires the belief to be $(0, 1)$, in which case C is not optimal for player 2.
If player 2 chooses S, then player 1's optimal action is L, in which case the consistency of player 2's belief requires the belief to be $(1, 0)$, in which case S is not optimal for player 2.
So the game has no pure strategy WSE.

(b) [5] Find a pure strategy Nash equilibrium of the game that is not the strategy profile for any weak sequential equilibrium. (Explain why the strategy profile is not part of a weak sequential equilibrium.)

Solution: Strategy profile (L, S, L). The game has no weak sequential equilibrium with this strategy profile because whatever is player 3's belief at her information set, R is better for her than L.