Economics 2030

Fall 2018 Martin J. Osborne

Solutions to Problem Set 11

1. We have $(v_1, v_2) = (1, 1)$, so that the payoff of player 1 in every subgame perfect equilibrium is at least 1. Since player 2's payoff always exceeds player 1's payoff we conclude that player 2's payoff in any subgame perfect equilibria exceeds 1. The path $((A, A), (A, A), \ldots)$ is not a subgame perfect equilibrium outcome path because player 2 can deviate to D, achieving a payoff of 5 in the first period and more than 1 in the subsequent subgame, which is better for her than the constant sequence $(3, 3, \ldots)$.

2. (a) Suppose that player 1 uses the grim strategy. If player 2 does too then her payoff is x in every period, so her discounted average payoff is x.

   If player 2 chooses D in any period she obtains y in that period and 1 subsequently, so that her discounted average payoff is $(1 - \delta)y + \delta$.

   Thus the strategy pair in which both players use the grim strategy is a Nash equilibrium if and only if $x \geq (1 - \delta)y + \delta$, or

$$\delta \geq \frac{y - x}{y - 1}.$$

   (b) Suppose that player 2 adheres to *tit-for-tat*. The action she takes in any period depends only on the outcome in the previous period, so we can group the subgames according to the outcome in the last period of the history that precedes them.

   (C, C) If player 1 adheres to *tit-for-tat* the outcome is (C, C) in every period, so that her discounted average payoff in the subgame is x. If she chooses D in the first period of the subgame, then adheres to *tit-for-tat*, the outcome alternates between (D, C) and (C, D), and her discounted average payoff is $y/(1 + \delta)$. Thus we need $x \geq y/(1 + \delta)$, or $\delta \geq (y - x)/x$, for a one-period deviation from *tit-for-tat* not to be profitable for player 1.

(C, D) If player 1 adheres to *tit-for-tat* the outcome alternates between (D, C) and (C, D), so that her discounted average payoff is $y/(1+\delta)$. If she deviates to C in the first period of the subgame, then adheres to *tit-for-tat*, the outcome is (C, C) in every period, and her discounted average payoff is x. Thus we need $y/(1+\delta) \geq x$, or $\delta \leq (y-x)/x$, for a one-period deviation from *tit-for-tat* not to be profitable for player 1.

(D, C) If player 1 adheres to *tit-for-tat* the outcome alternates between (C, D) and (D, C), so that her discounted average payoff is $\delta y/(1+\delta)$. If she deviates to D in the first period of the subgame, then adheres to *tit-for-tat*, the outcome is (D, D) in every period, and her discounted average payoff is 1. Thus we need $\delta y/(1+\delta) \geq 1$, or $\delta \geq 1/(y-1)$, for a one-period deviation from *tit-for-tat* not to be profitable for player 1.

(D, D) If player 1 adheres to *tit-for-tat* the outcome is (D, D) in every period, so that her discounted average payoff is 1. If she deviates to C in the first period of the subgame, then adheres to *tit-for-tat*, the outcome alternates between (C, D) and (D, C), and her discounted average payoff is $\delta y/(1+\delta)$. Thus we need $1 \geq \delta y/(1+\delta)$, or $\delta \leq 1/(y-1)$, for a one-period deviation from *tit-for-tat* not to be profitable for player 1.

The same arguments apply to deviations by player 2, so we conclude that (*tit-for-tat*, *tit-for-tat*) is a subgame perfect equilibrium if and only if $\delta = (y-x)/x$ and $\delta = 1/(y-1)$, or $y - x = 1$ and $\delta = 1/x$.

3. Consider an arbitrary subgame. Denote by $\hat{p}$ the minimum of p^m and the lowest price charged by either firm in the history preceding the subgame. (Note the necessity of considering an arbitrary subgame. It is not sufficient to consider only deviations at the start of the game.)

   If each firm i follows the strategy s_i, the outcome is the price pair $(\hat{p}, \hat{p})$ in every subsequent period.

   Suppose that firm 1 deviates from s_1 in the first period of the subgame by choosing p', then subsequently adheres to s_1.

   $p' > \hat{p}$ Firm 1 obtains zero profit in the first period of the subgame and the same profit in every subsequent period as it does if it does not deviate. Thus this deviation is not profitable.

$p' < \hat{p}$ If p' is close enough to $\hat{p}$, firm 1's profit is larger in the first period of the subgame than it is if it follows its strategy. Denote the increase in its profit by α. Subsequently it obtains half of the profit available at the price p', which is less than half of the monopoly profit. Denote the decrease in its profit in each period by β. The change in its discounted average payoff is $(1-\delta)\alpha - \delta\beta$, which is negative if $\delta > \alpha/(\alpha+\beta)$. Thus if δ is close enough to 1, the deviation is not profitable.

We conclude that the strategy pair is a subgame perfect equilibrium if δ is close enough to 1.

4. (a) Let $s_i : \{A, B\} \to \{C, D\}$ denote the (Markov) strategy of player i. Note first that, since actions in state B do not affect future states, in any MPE, the pair of actions chosen in state B must be a NE of the strategic game. Thus if $a < 0$, it must be that $s_1(B) = s_2(B) = D$, and if $a \geq 0$ then either $s_1(B) = s_2(B) = D$ or $s_1(B) = s_2(B) = C$. Similarly, if $s_{-i}(A) = D$, then it must be that $s_i(A) = D$. It follows that, in each state, the two players choose identical actions in any MPE.

   Suppose $s_1(A) = s_2(A) = s_1(B) = s_2(B) = D$. Consider one-shot deviations. Since, in this case, unilateral deviations do not affect the future states, there is no incentive to deviate in either state. Therefore, this strategy profile forms a MPE for every a and δ.

   Suppose $s_1(A) = s_2(A) = D$ and $s_1(B) = s_2(B) = C$. There is no incentive to deviate in state A, and in state B there is an incentive to deviate if and only if $a < 0$. Therefore, this strategy profile forms a MPE for every $a \geq 0$ and every δ.

   Suppose $s_1(A) = s_2(A) = C$ and $s_1(B) = s_2(B) = D$. There is no incentive to deviate in state B. In state A, adhering to this strategy gives payoff $3 + 3\delta + 3\delta^2 + \cdots$. A one-shot deviation to D gives payoff $4 + \delta + 3\delta^2 + 3\delta^3 + \cdots$. Therefore, this strategy profile forms a MPE for all a and all δ satisfying $3 + 3\delta \geq 4 + \delta$, that is, $\delta \geq 1/2$.

   Suppose $s_1(A) = s_2(A) = C$ and $s_1(B) = s_2(B) = C$. There is an incentive to deviate in state B if and only if $a < 0$. In state A, the argument is the same as in the last case except that the payoff from a one-shot deviation is now $4 + a\delta + 3\delta^2 + 3\delta^3$, implying that this strategy profile forms a MPE for all $a \in [0,2]$ and $\delta \geq 1/(3-a)$.

(b) The two sets are not the same. For example, one SPE is for both players to choose C at any history where neither has ever played D, and to choose D otherwise. There are also SPEs whose outcomes differ from the outcome of any MPE. For example, it is an SPE for player 1 to play C and player 2 to play D in the first period, then, in every subsequent period, both play C if neither has ever deviated from the equilibrium strategy in the past and D otherwise. The outcome of this SPE is $((C,D),(C,C),(C,C),\dots)$, whereas in any MPE the outcome has both players choosing the same action in every period. Each of these can be verified by considering one-shot deviations.