Economics 2030

Fall 2018 Martin J. Osborne

Solutions to Problem Set 10

1. If a player adheres to the strategy her payoff is x in every period. If she deviates from the strategy she obtains y and then at most 1 for k periods; then the situation is the same as it was at the beginning of the game. Thus she does not benefit from deviating if and only if

$$x(1-\delta^{k+1})/(1-\delta) \geq y + \delta(1-\delta^k)/(1-\delta),$$

or

$$x(1-\delta^{k+1}) \geq y(1-\delta) + \delta(1-\delta^k),$$

or

$$(x-1)\delta^{k+1} + (1-y)\delta + y - x \leq 0.$$

2. Suppose that player 1 adheres to the strategy. Then if player 2 can gain by deviating then she can gain by choosing D in the first period. If she does so, then player 1 chooses D in the second period, and continues to choose D until player 2 reverts to C. Thus player 2 has two options: she can revert to C, in which case in the next period she faces the same situation as she did at the start of the game, or she can continue to choose D, in which case player 1 will continue to do so too. We conclude that if player 2 can increase her payoff by deviating then she can do so either by alternating between D and C or by choosing D in every period.

   If she alternates between D and C then her stream of payoffs is $(3, 0, 3, 0, \ldots)$, with a discounted average of $(1-\delta) \cdot 3/(1-\delta^2) = 3/(1+\delta)$, while if she chooses D in every period her stream of payoffs is $(3, 1, 1, \ldots)$, with a discounted average of $3(1-\delta) + \delta = 3 - 2\delta$.

   Thus for the strategy pair to be a Nash equilibrium we need

$$\begin{aligned} a &\geq \frac{3}{1+\delta} \\ a &\geq 3 - 2\delta, \end{aligned}$$

or $\delta \geq \max\{(3-a)/2, (3-a)/a\}$.

There are values of $\delta < 1$ that satisfy these two equations only if $(3-a)/2 < 1$ and $(3-a)/a < 1$, or $a > \frac{3}{2}$.

Conclusion: if $a > \frac{3}{2}$ then for any $\delta \geq \max\{(3-a)/2, (3-a)/a\}$ the strategy pair is a Nash equilibrium. If $a \leq \frac{3}{2}$ there is no value of δ such that the strategy pair is a Nash equilibrium.

3. (a) Suppose that firm i uses the strategy s_i. If the other firm, j, uses s_j, then its discounted average payoff is

$$(1-\delta)\left(\tfrac{1}{2}\pi(p^m) + \tfrac{1}{2}\delta\pi(p^m) + \cdots\right) = \tfrac{1}{2}\pi(p^m).$$

If, on the other hand, firm j deviates to a price p then the closer this price is to p^m, the higher is j's profit, because the punishment does not depend on p. Thus by choosing p close enough to p^m the firm can obtain a profit as close as it wishes to $\pi(p^m)$ in the period of its deviation. Its profit during its punishment in the following k periods is zero. Once its punishment is complete, it can either revert to p^m or deviate once again. If it can profit from deviating initially then it can profit by deviating once its punishment is complete, so its maximal profit from deviating is

$$(1-\delta)\left(\pi(p^m) + \delta^{k+1}\pi(p^m) + \delta^{2k+2}\pi(p^m) + \cdots\right) = \frac{(1-\delta)\pi(p^m)}{1-\delta^{k+1}}.$$

Thus for (s_1, s_2) to be a Nash equilibrium we need

$$\frac{1-\delta}{1-\delta^{k+1}} \leq \tfrac{1}{2},$$

or

$$\delta^{k+1} - 2\delta + 1 \leq 0.$$

(b) Suppose that firm i uses the strategy s_i. If the other firm does so then its discounted average payoff is $\frac{1}{2}\pi(p^m)$, as in part *a*. If the other firm deviates to some price p with $c < p < p^m$ in the first period, and maintains this price subsequently, then it obtains

$\pi(p)$ in the first period and shares $\pi(p)$ in each subsequent period, so that its discounted average payoff is

$$
\begin{aligned}
(1-\delta)\big(\pi(p) + \tfrac{1}{2}\delta\pi(p) + \tfrac{1}{2}\delta^2\pi(p) + \cdots\big) \\
&= (1-\delta)\left(\pi(p) + \frac{\frac{1}{2}\delta\pi(p)}{1-\delta}\right) \\
&= (1-\delta)\pi(p) + \tfrac{1}{2}\delta\pi(p) \\
&= \tfrac{1}{2}(2-\delta)\pi(p).
\end{aligned}
$$

If p is close to p^m then $\pi(p)$ is close to $\pi(p^m)$ (because π is continuous). In fact, for any $\delta < 1$ we have $2 - \delta > 1$, so that we can find $p < p^m$ such that $(2-\delta)\pi(p) > \pi(p^m)$. Hence the strategy pair is not a Nash equilibrium of the infinitely repeated game for any value of δ.