Economics 2030

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Solutions to Problem Set 9

1. Let (U,d) be a bargaining problem, let α_i and β_i for $i = 1, 2$ be numbers with $\alpha_i > 0$, let

$$U' = \{(\alpha_1 u_1 + \beta_1, \alpha_2 u_2 = \beta_2) : (u_1, u_2) \in U\},$$

and let $d'_i = \alpha_i d_i + \beta_i$ for $i = 1, 2$. We have

$$\begin{aligned}(\alpha_1 u_1 + \beta_1 - (\alpha_1 d_1 + \beta_1))(\alpha_2 u_2 + \beta_2 - (\alpha_2 d_2 + \beta_2)) \\ = \alpha_1\alpha_2(u_1 - d_1)(u_2 - d_2)\end{aligned}$$

for all (u_1, u_2), so that (u_1^*, u_2^*) maximizes $(u_1 - d_1)(u_2 - d_2)$ over U if and only if $(\alpha_1 u_1^* + \beta_1, \alpha_2 u_2^* + \beta_2)$ maximizes $(v_1 - d'_1)(v_2 - d'_2)$ over U'.

2. (a) Suppose that $d = (0,0)$ and U is the triangle with corners at $(0,0)$, $(0,1)$ and $(1,0)$. The solution assigns to this problem the point $(\frac{1}{2}, \frac{1}{2})$. Now suppose that $d' = (0,0)$ and U' is the triangle with corners at $(0,0)$, $(0,1)$ and $(2,0)$. The solution assigns to this problem the point $(\frac{2}{3}, \frac{2}{3})$. But $U' = \{(2u_1, u_2) : (u_1, u_2) \in U\}$ and $d' = (2d_1, d_2)$, so INV requires, given the bargaining solution of (U,d), that the bargaining solution of (U', d') be $(1, \frac{1}{2})$.

   (b) Suppose that $d = (0,0)$ and U is the triangle with corners at $(0,0)$, $(0,1)$ and $(1,0)$. The solution assigns to this problem the point $(\frac{1}{2}, \frac{1}{2})$. Now suppose that $d' = (0,0)$ and U' is the quadrilateral with corners at $(0,0)$, $(0, \frac{1}{2})$, $(\frac{1}{2}, \frac{1}{2})$, and $(1,0)$. The solution assigns to this problem the point $(\frac{2}{3}, \frac{1}{3})$. But $d = d'$, $U' \subset U$, and the solution of (U,d) is in U', so IIA requires that the solution of (U', d) be the same as the solution of (U,d).

3. Satisfies INV, SYM, and PAR, but not IIA.

   Argument for INV: Denote the original problem by (U,d) and its solution by z. Transform the payoffs of player i by $u_i \mapsto \alpha_i u_i + \beta_i$ for

$i = 1, 2$ and denote the transformed problem by (U', d'). We need to show that $(\alpha_1 z_1 + \beta_1, \alpha_2 z_2 + \beta_2)$ is the solution of (U', d').

Because the transformations are increasing, $\alpha_i x_i^* + \beta_i$ is the maximal payoff of player i in the transformed problem. Denote this payoff x_i^{**}. Let λ be such that $z = \lambda d + (1 - \lambda)x^*$. We have

$$\begin{aligned} \alpha_i z_i + \beta_i &= \alpha_i(\lambda d_i + (1 - \lambda)x_i^*) + \beta_i \\ &= \lambda(\alpha_i d_i + \beta_i) + (1 - \lambda)(\alpha_i x_i^* + \beta_i) \\ &= \lambda d_i' + (1 - \lambda)x_i^{**}. \end{aligned}$$

That is, $(\alpha_1 z_1 + \beta_1, \alpha_2 z_2 + \beta_2)$ is on the line joining d' and x^{**} for the transformed problem. It is on the Pareto frontier of U' because z is on the Pareto frontier of U and the transformations are increasing. Thus $(\alpha_1 z_1 + \beta_1, \alpha_2 z_2 + \beta_2)$ is the solution of (U', d').

Argument for SYM: If a problem is symmetric, $x_1^* = x_2^*$, so the solution is symmetric.

Argument for PAR: By definition the solution is Pareto efficient.

Argument against IIA: Let $(d_1, d_2) = (0, 0)$. The solution of (U, d), where U is the convex hull of $(0, 0)$, $(1, 0)$, and $(0, 1)$ is $(\frac{1}{2}, \frac{1}{2})$. This solution is a member of the convex hull of $(0, 0)$, $(\frac{1}{2}, \frac{1}{2})$, and $(0, 1)$, but the solution of this second problem is not $(\frac{1}{2}, \frac{1}{2})$.

4. The Nash bargaining solution maximizes

$$(f(\ell^*) - \ell^* w)(\ell^* w + (L - \ell^*)w_0 - L w_0),$$

or

$$(f(\ell^*) - \ell^* w)\ell^*(w - w_0).$$

This function is a quadratic in w that is equal to zero when $w = f(\ell^*)/\ell^*$ and when $w = w_0$. Thus the value of w that maximizes it is

$$\tfrac{1}{2}(f(\ell^*)/\ell^* + w_0),$$

the average of the average output of a worker and the "outside wage" w_0.