**Economics 2030**

Winter 2018 | Martin J. Osborne

**Solutions to Problem Set 12**

1. At the initial history choose $A$ and $B$ each with probability $\frac{1}{2}$; at the second information set choose $\ell$.

2. First note for player 1, $C$ strictly dominates $N$ after the history $E$, so in any weak sequential equilibrium she chooses $C$ after the history $E$.

   Next notice that there is no equilibrium in which player 1 chooses $N$ after the history $I$. In any such equilibrium, player 2 believes that the history was $(E, C)$ and hence chooses $D$. However, when player 2 uses such a strategy, player 1 prefers $C$ to $N$ after the history $I$.

   Now consider the possibility of an equilibrium in which player 1 chooses $C$ in both cases. In such an equilibrium, player 2's belief at her information set must assign probability $\frac{1}{2}$ to both $(E, C)$ and $(I, C)$, so that $A$ is optimal. But if player 2 chooses $A$, player 1 prefers $N$ after history $I$. Thus there is no such equilibrium.

   The remaining possibility is that player 1 randomizes after the history $I$. In any such equilibrium, player 1 is indifferent between $N$ and $C$, which requires player 2 to choose $A$ with probability $\frac{1}{2}$. For player 2 to randomize, she must be indifferent between $A$ and $D$. If she believes that the history was $(E, C)$ with probability $p$, indifference requires

   $$-50p + 150(1 - p) = 0,$$

   or $p = \frac{3}{4}$. If player 1 chooses $C$ with probability $q$ after the history $I$ then player 2's belief assigns probability $\frac{3}{4}$ to $(E, C)$ if

   $$\frac{3}{4} = \frac{\frac{1}{2}}{\frac{1}{2} + \frac{1}{2}q},$$

   or $q = \frac{1}{3}$.

   Thus the game has a single weak sequential equilibrium, in which

- player 1 chooses $C$ after the history $E$ and chooses $C$ with probability $\frac{1}{3}$ after the history $I$
- player 2 chooses $A$ with probability $\frac{1}{2}$ and believes that the history was $(E, C)$ with probability $\frac{3}{4}$.

3. Denote the strategy of player 1 by $(\alpha, \beta, \gamma)$.

    - If $\beta > \gamma$ then by the weak consistency of player 2's belief she assigns higher probability to $M$ than to $R$, so that her optimal action is $L$. Thus player 1's optimal action is $M$. The assessment $(((0,1,0),(1,0)),(1,0))$ is thus a weak sequential equilibrium. It is also a sequential equilibrium, because for the strategy $(\varepsilon, 1-\varepsilon, \varepsilon)$ of player 1, the belief implied by Bayes' Law at player 2's information set is close to $(1,0)$.
    - If $\beta < \gamma$ then by the weak consistency of player 2's belief she assigns higher probability to $R$ than to $M$, so that her optimal action is $R$. Thus player 1's optimal action is $L$, contradicting $\gamma > 0$. Hence the game has no weak sequential equilibrium in which $\beta < \gamma$.
    - If $\beta = \gamma > 0$ then by the weak consistency of player 2's belief she assigns the same probabilities to $M$ and $R$, and is thus indifferent between $L$ and $R$. Denote by $q$ the probability that she chooses $L$. To make player 1 indifferent between $M$ and $R$ we need

      $$3q - 2(1-q) = 2q - (1-q),$$

      or $q = \frac{1}{2}$. For this strategy of player 2, player 1's expected payoff to $M$ (and $R$) is $\frac{1}{2}$. Her payoff to $L$ is 1, so there is no weak sequential equilibrium of this type.
    - If $\beta = \gamma = 0$ (i.e. player 1 chooses $L$ with probability 1) then weak consistency puts no restriction on player 2's belief that the history is $M$. Denote the value $p$.

      If $p > \frac{1}{2}$, then player 2's optimal action is $L$, in which case $M$ is better for player 1 than $L$, so there is no weak sequential equilibrium in which $p > \frac{1}{2}$.

      If $p < \frac{1}{2}$, then player 2's optimal action is $R$, and hence $L$ is optimal for player 1. Thus the game has a weak sequential equilibrium $((1,0,0),(0,1),(p,1-p))$ for any $p < \frac{1}{2}$. The beliefs $(p, 1-p)$ are derived by Bayes' Law from strategies of player 1 of the form $(1-\varepsilon, p\varepsilon, (1-p)\varepsilon)$, so any such assessment is also a sequential equilibrium.

If $p = \frac{1}{2}$, then player 2 is indifferent between $L$ and $R$. Denote the probability that player 2 chooses $L$ by $q$. For $L$ to be optimal for player 1 we require $1 \geq 3q - 2(1-q)$ and $1 \geq 2q - (1-q)$, or $q \leq \frac{3}{5}$. By the argument for the previous case, the associated assessment is a sequential equilibrium.

We conclude that the sequential equilibria of the game are the assessments $(((1,0,0),(q,1-q)),(\frac{1}{2},\frac{1}{2}))$ with $0 \leq q \leq \frac{3}{5}$, the assessments $(((1,0,0),(0,1)),(p,1-p))$ with $0 \leq p \leq \frac{1}{2}$, and the assessment $(((0,1,0),(1,0)),(1,0))$.

4. In a weak sequential ("separating") equilibrium in which a strong challenger chooses *Ready* and a weak one chooses *Unready*, the incumbent's belief assigns probability 1 to the history (*Strong*, *Ready*) at her top information set and probability 1 to the history (*Weak*, *Unready*) at her bottom information set. Thus the incumbent chooses $A$ at the top information set and $F$ at the bottom one. Given these actions of the incumbent, the challenger's payoff decreases if she switches from $R$ to $U$ after the history *Strong*. For her payoff not to increase if she switches from $U$ to $R$ after the history *Weak* we need $a_1 \leq 3$. We conclude that the game has a weak sequential equilibrium in which the challenger chooses *Ready* after the history *Strong* and *Unready* after the history *Weak* if and only if $a_1 \leq 3$.

   If an assessment in which both types of challenger choose $U$ is a weak sequential equilibrium then at the incumbent's bottom information set she believes that the history is (*Strong*, *Unready*) with probability $p$ and (*Weak*, *Unready*) with probability $1-p$. Thus the incumbent's action at her bottom information set is $F$ if $p < \frac{1}{4}$, $A$ if $p > \frac{1}{4}$, and any mixture of $A$ and $F$ if $p = \frac{1}{4}$.

   Now consider the incumbent's action at her top information set. In a weak sequential equilibrium in which the challenger chooses $U$ after both the history *Strong* and the history *Weak*, the incumbent's belief at her top information set is not restricted, because this information set is not reached with positive probability. If $a_2 > b_2$ then $A$ is the unique optimal action regardless of the incumbent's belief, whereas if $a_2 \leq b_2$ then $F$ is optimal if the probability the incumbent assigns to (*Strong*, *Ready*) is small enough.

Consider each case in turn.

$p < \frac{1}{4}$ For the challenger not to be able to profitably deviate after the history *Strong*, we need the incumbent to assign probability of at least $\frac{1}{2}$ to $F$ at her top information set, which requires $a_2 \leq b_2$. Denote the probability that the incumbent assigns to $A$ at her top information set by $\pi$. Then for the assessment to be a weak sequential equilibrium we need $\pi a_1 + (1 - \pi) b_1 \leq 3$. Thus for the game to have a weak sequential equilibrium in which both types of challenger choose $U$, we need $a_2 \leq b_2$ and $\pi a_1 + (1 - \pi) b_1 \leq 3$ for some $\pi \leq \frac{1}{2}$. (If $a_2 \leq b_2$ then the incumbent's belief at her top information set may be chosen to induce any value of $\pi$.)

We conclude that if $p < \frac{1}{4}$ then the game has a weak sequential equilibrium in which both types of challenger choose $U$ if and only if $a_2 \leq b_2$ and either (*a*) $a_1 \geq b_1$ and $b_1 \leq 3$ or (*b*) $a_1 \leq b_1$ and $\frac{1}{2}(a_1 + b_1) \leq 3$.

$p > \frac{1}{4}$ In this case the challenger cannot profitably deviate after the history *Strong*, regardless of the incumbent's action at her top information set.

$a_2 > b_2$ The incumbent chooses $A$ at her top information set regardless of her belief, so the challenger cannot profitably deviate after the history *Weak* if and only if $a_1 \leq 5$.

$a_2 \leq b_2$ In this case there are beliefs under which any mixture of $A$ and $F$ is optimal for the incumbent at her top information set. Thus the challenger cannot profitably deviate after the history *Strong* if and only if $\min\{a_1, b_1\} \leq 5$.

We conclude that if $p > \frac{1}{4}$, then the game has a weak sequential equilibrium in which both types of challenger choose $U$ if and only if either (*a*) $a_2 > b_2$ and $a_1 \leq 5$, or (*b*) $a_2 \leq b_2$ and $\min\{a_1, b_1\} \leq 5$.

$p = \frac{1}{4}$ In this case both $A$ and $F$ (and any mixture of them) are optimal for the incumbent at bottom information set. The action $A$ yields the challenger more than $F$ does, so the game has a weak sequential equilibrium in which both types of challenger choose $U$ if and only if the conditions for the case $p > \frac{1}{4}$ are satisfied.