ECO2030: Microeconomic Theory II, module 1

Lecture 12

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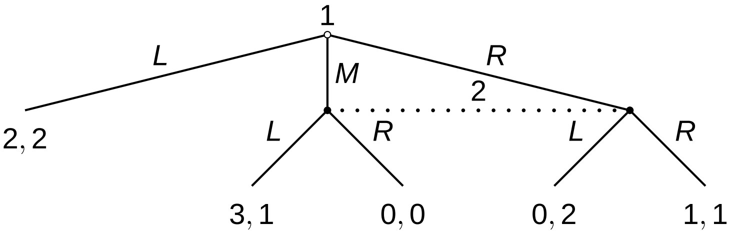
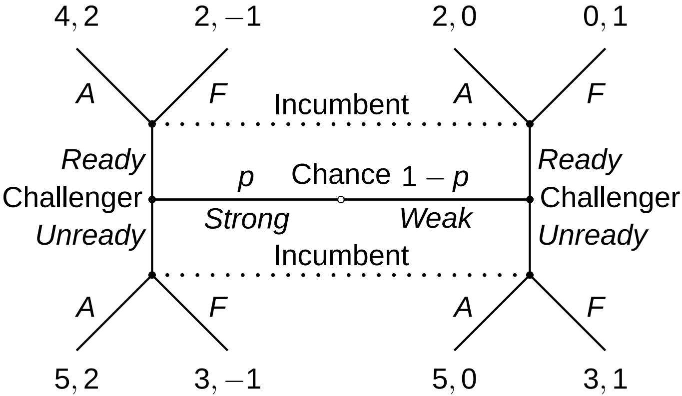
Extensive games with imperfect information

- Extensive game with perfect information: players perfectly informed about *past* actions
- Now consider games in which players are *not* perfectly informed about past actions
- Example:



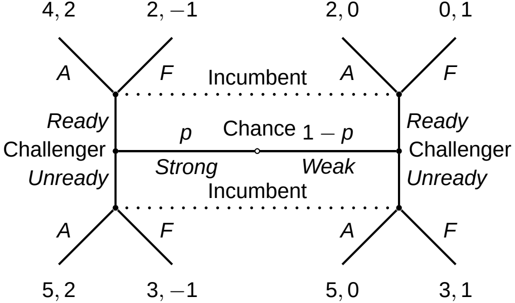
Extensive games with imperfect information

- Example with chance move:



Extensive games with imperfect information

- To specify information, introduce collection $(\mathcal{I}_i)_{i \in N}$ of *information partitions*
- $\mathcal{I}_i$ is a partition of the histories after which i moves
- Each member of the partition is an *information set*
- If $h \in I_i$ and $h' \in I_i$, with $I_i \in \mathcal{I}_i$, then i cannot distinguish between h and h'



$$\mathcal{I}_{\text{Incumbent}} = \{\{(Strong, Ready), (Weak, Ready)\}, \{(Strong, Unready), (Weak, Unready)\}\}$$

Extensive games with imperfect information

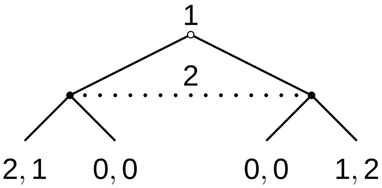
Definition

An extensive game consists of

- A finite set N (set of *players*)
- A set H of histories
- A function $P : H \setminus Z(H) \to N \cup \{c\}$ (*player function*, specifying player or chance, c, who moves after each nonterminal history)
- A function f_c that associates with every history h for which $P(h) = c$ a probability measure $f_c(\cdot|h)$ on $A(h)$, with each such probability measure independent of every other one
- For each player $i \in N$ a partition $\mathcal{I}_i$ of $\{h \in H : P(h) = i\}$ with $A(h) = A(h')$ whenever h and h' are in the same member of the partition (i's *information partition*)
- For each player $i \in N$ a preference relation $\succsim_i$ on lotteries over $Z(H)$ represented by expected value of payoff function

Extensive games with imperfect information

Example

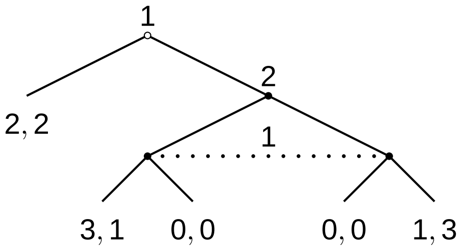


Models same situation as

- strategic game in which players 1 and 2 choose actions simultaneously
- extensive game with perfect information and simultaneous moves in which players 1 and 2 move simultaneously

Extensive games with imperfect information

Example

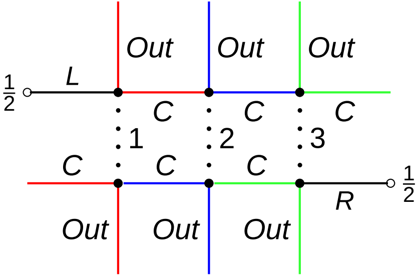


Models same situation as

- extensive game with perfect information and simultaneous moves in which player 1 moves and then players 1 and 2 move simultaneously

Extensive games with imperfect information

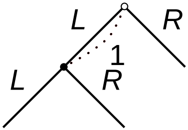
Example



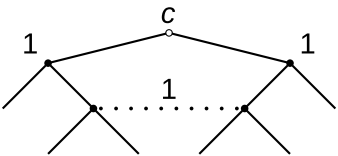
- Player 1 does not know whether she is the first mover or whether she is moving after the other players have moved
- Player 2 does not know whether she is moving after player 1 and before player 3, or the other way around
- Player 3 does not know whether she is the first mover or whether she is moving after the other players have moved

Extensive games with imperfect information

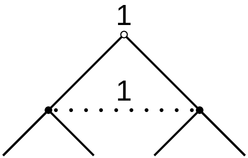
Examples



Player does not know whether she is choosing action at start of game, or whether she has already chosen an action



When choosing at her last information set, player does not know move of chance, which she knew at start of game



When making her first choice, player does not know action she chose at start of game

These games have *imperfect recall*

Extensive games with imperfect information

Perfect recall

- Game has perfect recall if at every point every player remembers whatever she knew in the past
- Will restrict throughout to games with perfect recall

Strategies

Denote by $A(I_i)$ the set of actions available to player i at the information set I_i

- Remember that for every history in I_i the set of actions is the same

Definition

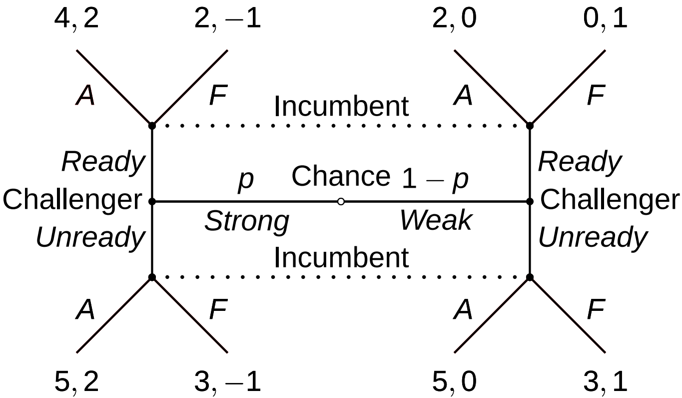
A pure strategy of player $i \in N$ in an extensive game $\langle N, H, P, f_c, (\mathcal{I}_i), (\succsim_i)\rangle$ is a function that assigns an action in $A(I_i)$ to each information set $I_i \in \mathcal{I}_i$

$\Rightarrow$ number of pure strategies of player i = product of numbers of actions at information sets of player i

- Given set of strategies for each player, can define strategic form of extensive game as before

Strategies

Example

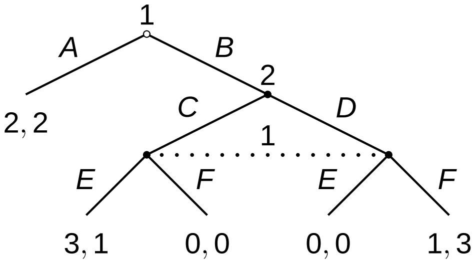


Pure strategy of incumbent specifies actions as each of her two information sets, so 4 pure strategies: *AA* (i.e. *A* at each information set), *AF* (i.e. *A* at bottom information set, *F* at top one), *FA*, *FF*

Mixed strategies

A mixed strategy of player i is a probability measure over player i's set of pure strategies

Example

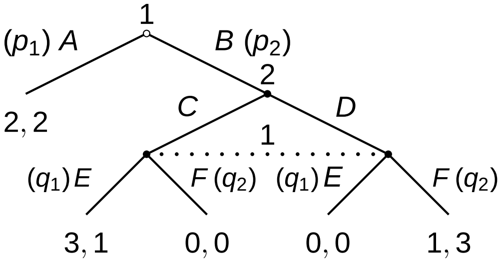


- Player 1's pure strategies: AE, AF, BE, BF
- Mixed strategy of player 1 is probability distribution (p_1, p_2, p_3, p_4) over these four strategies $(p_1 + p_2 + p_3 + p_4 = 1)$

Behavioral strategies

A behavioral strategy of player i is a collection $(\beta_i(I_i))_{I_i \in \mathcal{I}_i}$ of independent probability measures, where $\beta_i(I_i)$ is a probability distribution over $A(I_i)$

Example



Behavioral strategy of player 1 is pair $((p_1, p_2), (q_1, q_2))$:

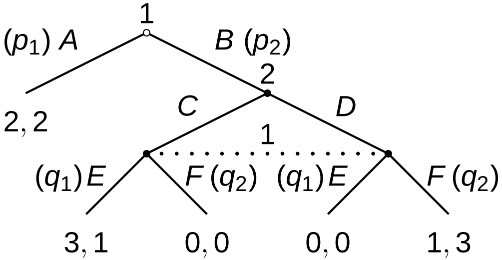
- p_1 and p_2 are probabilities assigned to A and B at start of game ($p_1 + p_2 = 1$)
- q_1 and q_2 are probabilities assigned to E and F at player 1's second information set ($q_1 + q_2 = 1$)

Mixed and behavioral strategies

- Mixed and behavioral strategies are different ways of formulating a player's randomization
- However, they are closely related
- Say two (mixed or behavioral) strategies of a player are outcome-equivalent if for every collection of pure strategies of the other players the two strategies induce the same probability distribution over terminal histories

Mixed and behavioral strategies

Example

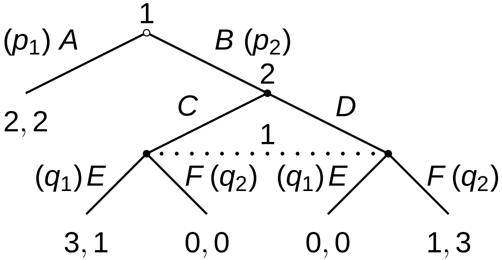


Claim: For every behavioral strategy of player 1 there is a mixed strategy that is outcome-equivalent

Behavioral		Mixed
$A : p_1$		$AE : p_1 q_1$
$B : p_2$	$\Rightarrow$	$AF : p_1 q_2$
$E : q_1$		$BE : p_2 q_1$
$F : q_2$		$BF : p_2 q_2$

Mixed and behavioral strategies

Example



Claim: For every mixed strategy of player 1 there is a behavioral strategy that is outcome-equivalent

Mixed		Behavioral
$AE : r_1$		$A : r_1 + r_2$
$AF : r_2$	$\Rightarrow$	$B : r_3 + r_4$
$BE : r_3$		$E : r_3/(r_3 + r_4)$
$BF : r_4$		$F : r_4/(r_3 + r_4)$

Mixed and behavioral strategies

Outcome-equivalence of mixed and behavioral strategies holds for all finite games with perfect recall

Proposition

Let Γ be a finite extensive game with perfect recall.

- For any behavioral strategy of a player in Γ there is an outcome-equivalent mixed strategy.
- For any mixed strategy of a player in Γ there is an outcome-equivalent behavioral strategy.

Subsequently we restrict attention to games with perfect recall and work with behavioral strategies

Nash equilibrium

- For any profile of mixed strategies, let $O(\sigma)$ be the outcome of σ: the probability distribution over terminal histories generated by σ

Definition

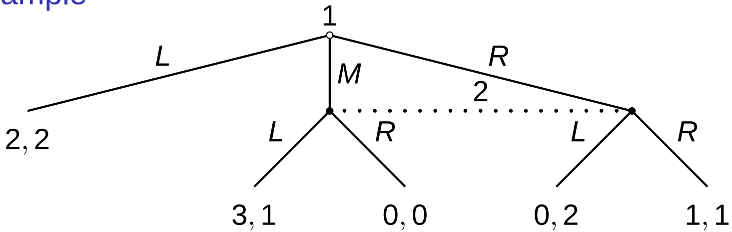
A Nash equilibrium in mixed strategies of an extensive game $\langle N, H, P, f_c, (\mathcal{I}_i), (\succsim_i) \rangle$ is a profile σ of mixed strategies such that for all $i \in N$

$$O(\sigma^*_{-i}, \sigma^*_i) \succsim_i O(\sigma^*_{-i}, \sigma_i) \text{ for every mixed strategy } \sigma_i \text{ of player } i$$

A Nash equilibrium in behavioral strategies is defined similarly

Nash equilibrium

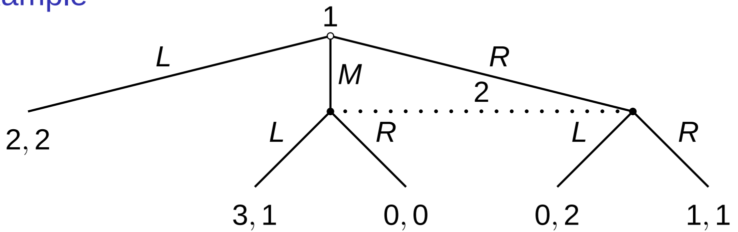
Example



- One Nash equilibrium: (M, L)

Nash equilibrium

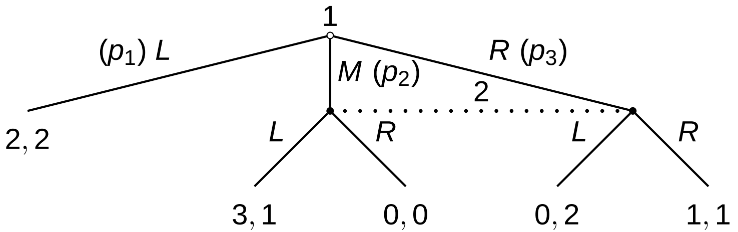
Example



- Another Nash equilibrium: (L, R)
  - But if player 1 deviates to M or R, player 2's action L is better than R *regardless of whether she believes player 1 chose M or R*
  - Like incumbent's action *Fight* in NE (*Out*, *Fight*) of Entry game, player 2's strategy R is not optimal if player 2's information set is reached
  - This NE *is* subgame perfect: the game has no proper subgame
  - We need new refinement of NE

Nash equilibrium

Example



- In this game, optimal action of player 2 is L regardless of her belief about whether player 1 chose M or R
- But for other payoffs, optimal action depends on her belief
- If player 1 chooses M and/or R with positive probability, player 2's belief can be derived from player 1's strategy
- But if player 1 chooses L, player 2's belief *cannot* be derived from player 1's strategy
  - Need to specify player 2's belief as part of equilibrium

Beliefs and assessments

A belief system for an extensive game is a function that assigns to every information set a probability measure over the set of histories in the information set

- Probability measure assigned to information set I represents beliefs of player $P(I)$ who moves at I about probabilities of histories in I
- For belief system μ, probability measure assigned to I is $\mu(I)$ and the probability this measure assigns to history h is $\mu(I)(h)$
- Restrict to games with perfect recall in which every information set contains finite number of histories

An assessment in an extensive game is pair (β, μ) where β is a profile of behavioral strategies and μ is a belief system

Equilibrium

Sequential rationality
Each player's strategy is optimal given her beliefs

Consistency of beliefs
The belief system is consistent with the strategy profile

Sequential rationality

An assessment (β, μ) is sequentially rational if for every player i and every information set $I_i \in \mathcal{I}_i$ the strategy β_i of player i is a best response to the other players' strategies β_{-i} given i's beliefs $\mu(I_i)$ at I_i

Equilibrium

- Consistency requirement has several possible formulations
- Following condition is very weak

Weak consistency

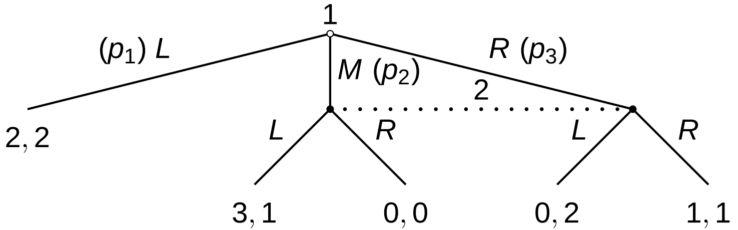
An assessment (β, μ) is weakly consistent if for every information set I_i reached with positive probability given β, the probability assigned by μ to each history h^* in I_i is given by Bayes' rule:

$$\mu(I_i)(h^*) = \frac{\Pr(h^* \text{ according to } \beta)}{\sum_{h \in I_i} \Pr(h \text{ according to } \beta)}$$

- Note that this condition imposes *no* restriction of beliefs at information sets not reached if players follow β

Equilibrium

Weak consistency



- An assessment in which player 1 chooses L and player 2 holds *any* belief at her information set is weakly consistent because given player 1's strategy, player 2's information set is not reached
- If $p_2 + p_3 > 0$ then weak consistency requires that player 2's belief assign probability $p_2/(p_2 + p_3)$ to M and probability $p_3/(p_2 + p_3)$ to R

Equilibrium

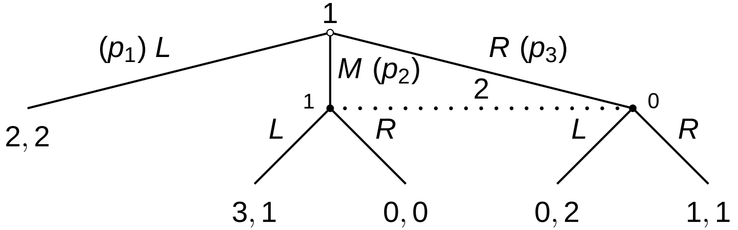
Definition

An assessment is a weak sequential equilibrium of a finite extensive game with perfect recall if it is sequentially rational and weakly consistent

- MWG use the term *weak perfect Bayesian equilibrium*

Equilibrium

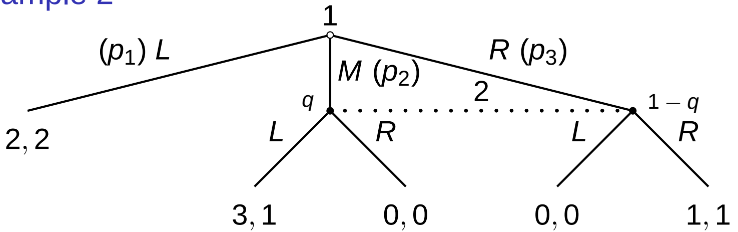
Example 1



- Start by looking at P2's choice
- For *any* belief at P2's information set, only L is optimal
- So in any WSE P2 chooses L
- Given that P2 chooses L, P1's optimal action is M
- What are P2's beliefs at her information set?
- Weak consistency $\Rightarrow q = 1$
- So unique WSE, with strategies (M, L) and beliefs $(1, 0)$

Equilibrium

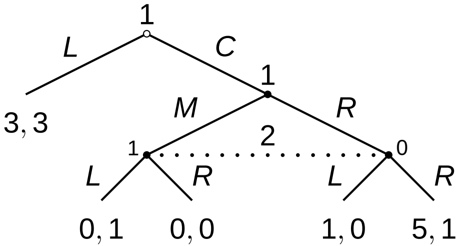
Example 2



- Start by looking at P2's choice
- If $q > \frac{1}{2}$ then L is only optimal action; if $q < \frac{1}{2}$ then R is only optimal action; if $q = \frac{1}{2}$ then both L and R are optimal
- If P2 chooses L then P1 chooses $M \Rightarrow$ beliefs $(1, 0) \Rightarrow L$ is optimal $\Rightarrow$ assessment $((M, L), (1, 0))$ is WSE
- If P2 chooses R then P1 chooses $L \Rightarrow$ beliefs unrestricted by weak consistency; need $q \leq \frac{1}{2}$ for R to be optimal $\Rightarrow$ any assessment $((L, R), (q, 1 - q))$ with $q \leq \frac{1}{2}$ is WSE

Equilibrium

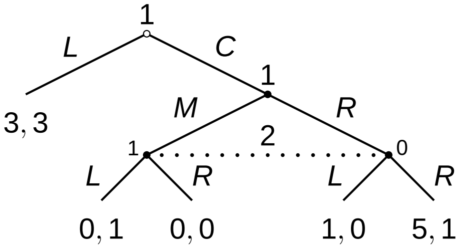
Example 3



- Consider assessment in which P1's strategy is (L, R), P2's strategy is L, and P2's belief is $(1, 0)$
- P1's strategy is optimal (payoffs to other strategies ≤ 1)
- P2's strategy is optimal given her belief
- P2's belief does not violate weak consistency because information set is not reached
- So assessment is WSE

Equilibrium

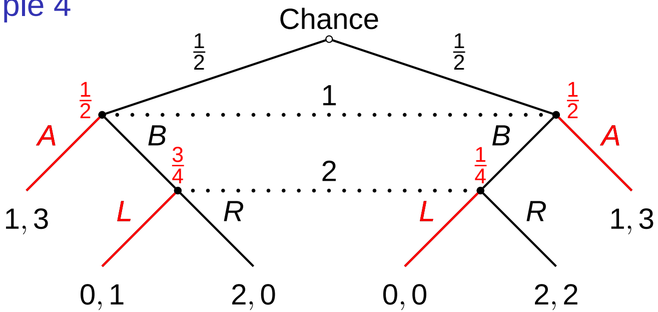
Example 3



- Is assessment an SPE?
- No! In subgame following C, L is not optimal for P2 given P1's strategy
- Problem is that P2's belief in the WSE isn't derived from P1's strategy *in the subgame*—weak consistency doesn't require it to be because the subgame is not reached if P1 follows her strategy

Equilibrium

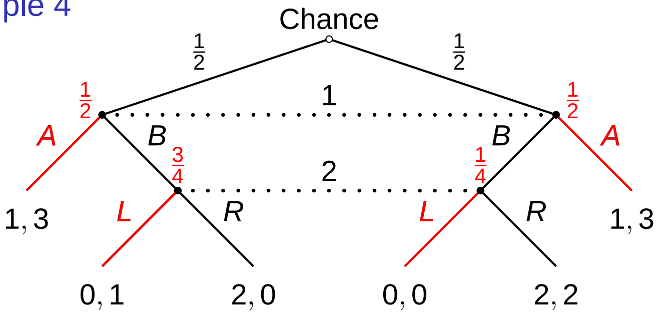
Example 4



- Consider indicated assessment
- P1's strategy is optimal given her belief
- P2's strategy is optimal given her belief (payoff to L is $\frac{3}{4} \cdot 1 + \frac{1}{4} \cdot 0 = \frac{3}{4}$, payoff to R is $\frac{3}{4} \cdot 0 + \frac{1}{4} \cdot 2 = \frac{1}{2}$)

Equilibrium

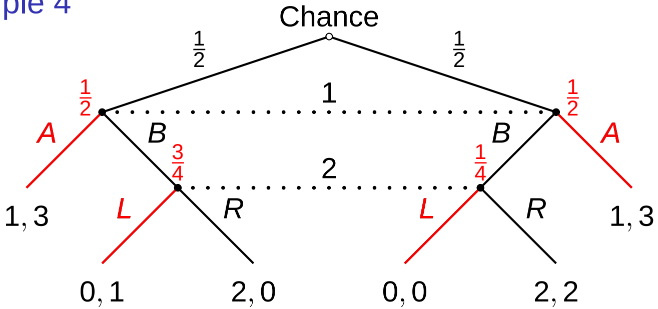
Example 4



- Belief at P1's information set is consistent with move of chance
- Belief at P2's information set does not violate weak consistency because information set is not reached
- So assessment is WSE

Equilibrium

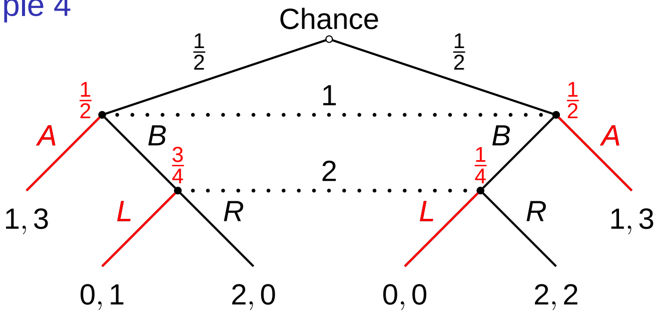
Example 4



- But belief at P2's information set cannot be derived from any *alternative* strategy of P1
  - If P1 uses strategy $(p, 1-p)$ with $0 \leq p < 1$ then belief at P2's information set is $(\frac{1}{2}, \frac{1}{2})$
- And for every belief at P2's information set that *is* derived from a strategy of P1, R is optimal for P2, so that B, not A, is optimal for P1

Equilibrium

Example 4



- *Conclusion*: Although assessment is WSE, it does not seem reasonable, and in no reasonable equilibrium does P1 choose *A*

Consistent belief system

- Examples 4 and 5 suggest we need to strengthen weak consistency and restrict beliefs at information sets not reached in equilibrium
- One possibility: require that there exist *some* sequence of assessments converging to (β, μ) in which every strategy β_i assigns positive probability to every action of player i and μ is derived from β by Bayes' rule

Definition

An assessment (β, μ) is consistent if there is a sequence $((\beta^n, \mu^n))_{n=1}^{\infty}$ of assessments that converges to (β, μ) in Euclidian space and has the properties that each strategy profile β^n is completely mixed and that each belief system μ^n is derived from β^n using Bayes' rule

Note: The strategy profiles β^n in the sequence are not required to be optimal with respect to any beliefs

Sequential equilibrium

Definition

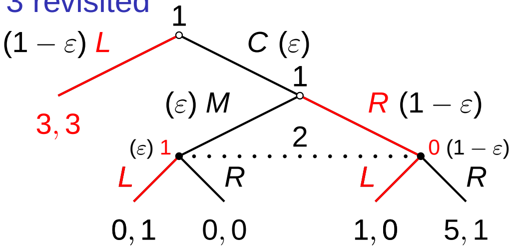
An assessment is a sequential equilibrium of a finite extensive game with perfect recall if it is sequentially rational and consistent

Proposition

- Every finite extensive game with perfect recall has a sequential equilibrium
- If (β, μ) is a sequential equilibrium then β is a Nash equilibrium
- In an extensive game with perfect information (β, μ) is a sequential equilibrium if and only if β is a subgame perfect equilibrium
- In any extensive game with perfect recall the strategy profile in any sequential equilibrium is a subgame perfect equilibrium

Sequential equilibrium

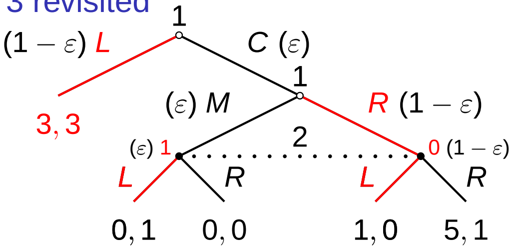
Example 3 revisited



- Is this assessment, say (β, μ), a sequential equilibrium?
- That is, is there a sequence of assessments in which the strategies are completely mixed and the beliefs are derived from strategies using Bayes' Law that converges to (β, μ)?
- Suppose P1's strategy is completely mixed and close to β_1
- Then P2's belief, by Bayes' Law, assigns probability ε to (C, M) and probability $1 - \varepsilon$ to (C, R)
- P2's optimal action given this belief is R, not L

Sequential equilibrium

Example 3 revisited

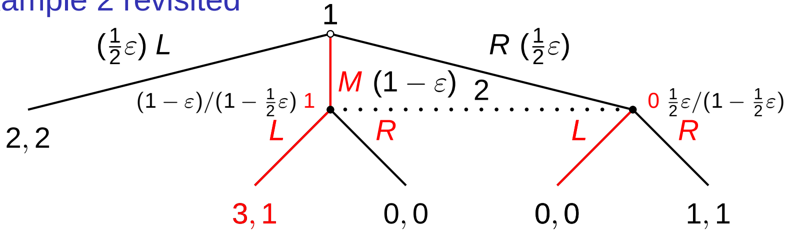


Conclusion

- No sequence of assessments in which strategies are completely mixed and beliefs are derived from strategies using Bayes' Law converges to (β, μ)
- So (β, μ) is *not* a sequential equilibrium

Equilibrium

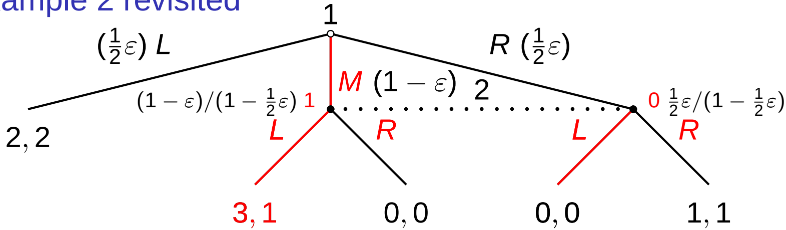
Example 2 revisited



- Is this assessment, say (β, μ), a sequential equilibrium?
- That is, is there a sequence of assessments in which the strategies are completely mixed and the beliefs are derived from strategies using Bayes' Law that converges to (β, μ)?
- Suppose P1's strategy is completely mixed and close to β_1
- Then P2's belief assigns probability $(1-\varepsilon)/(1-\varepsilon/2)$ to *M* and probability $(\varepsilon/2)/(1-\varepsilon/2)$ to *R*

Equilibrium

Example 2 revisited



- P2's best response to this belief is L
- Thus the sequence $(\beta^n, \mu^n)_{n=1}^{\infty}$ of assessments in which

$$\beta_1^n(\varnothing) = (\tfrac{1}{2}\varepsilon^n, 1-\varepsilon^n, \tfrac{1}{2}\varepsilon^n)$$

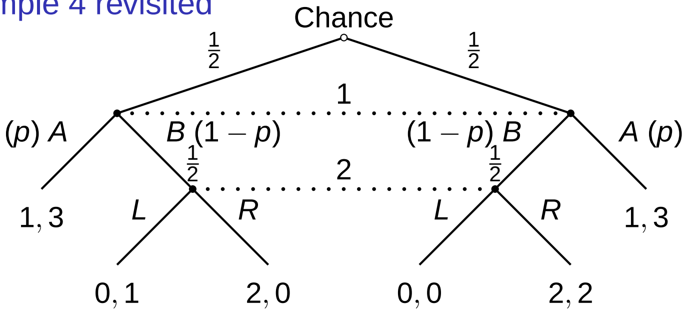
$$\beta_2^n(\{M, R\}) = (1-\varepsilon^n, \varepsilon^n)$$

$$\mu^n(\{M, R\}) = ((1-\varepsilon^n)/(1-\tfrac{1}{2}\varepsilon^n), \tfrac{1}{2}\varepsilon^n/(1-\tfrac{1}{2}\varepsilon^n))$$

  satisfies conditions in definition of sequential equilibrium
- So indicated assessment satisfies conditions $\Rightarrow$ assessment $((M, L), (1, 0))$ is sequential equilibrium

Equilibrium

Example 4 revisited



- For every strategy of P1 that assigns positive probability to *B*, belief at P2's information set derived by Bayes' Law is $(\frac{1}{2}, \frac{1}{2})$
- For this belief, only *R* is optimal for P2
- Given that P2 chooses *R*, only *B* is optimal for P1
- So in any sequential equilibrium, P1 chooses *B* (and not *A*)

Sequential equilibrium

Summary

- *Weak sequential equilibrium* requires strategies to be optimal given beliefs and beliefs to be derived from strategies at information sets reached with positive probability when players follow strategies
- Does not restrict beliefs at information sets not reached when players follow strategies
- Examples show that absence of restriction at such information sets leads to equilibria that seem unreasonable
- *Sequential equilibrium* imposes a restriction that rules out the unreasonable equilibria in the examples, although the meaning of the condition is not very clear