

ECO2030: Microeconomic Theory II,  
module 1  
Lecture 11

Martin J. Osborne

Department of Economics  
University of Toronto

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## Finitely repeated games

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## Dynamic games

# Nash equilibrium of infinitely repeated games

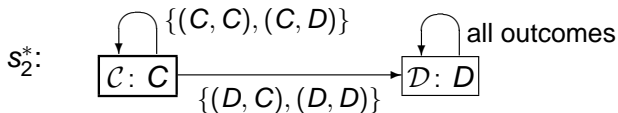
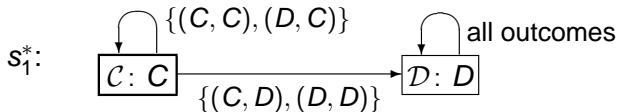
- ▶ When players are sufficiently patient, set of Nash equilibrium payoff profiles of infinitely repeated game with discounting is approximately equal to set of strictly enforceable payoff profiles of stage game
- ▶ Equilibrium strategies involve “punishments” for players who deviate from norm
- ▶ Are the Nash equilibria subgame perfect?

# SPE of infinitely repeated *Prisoner's Dilemma*

Consider Nash equilibrium of infinitely repeated *Prisoner's Dilemma*

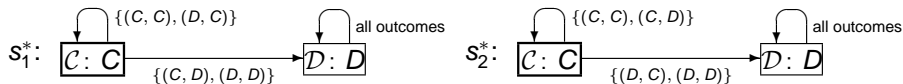
	C	D
C	3, 3	0, 4
D	4, 0	1, 1

in which players' strategies are



Is this strategy pair a SPE?

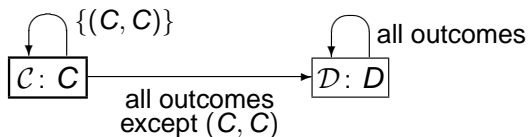
# SPE of infinitely repeated *Prisoner's Dilemma*



- ▶ Consider subgame following history  $(C, D)$
- ▶ Suppose P2 uses  $s_2^*$
- ▶ P1 uses  $s_1^*$  in subgame
  - ⇒ outcome path in subgame is  $((D, C), (D, D), (D, D), \dots)$
  - ⇒ payoff stream in subgame is  $4, 1, 1, \dots$  to P1, with discounted average  $4(1 - \delta) + \delta$
- ▶ P1 deviates in subgame to strategy that chooses C regardless of history
  - ⇒ outcome in subgame is  $(C, C)$  in every period
  - ⇒ discounted average payoff 3 to P1
- ▶ Better for P1 to deviate if  $3 > 4(1 - \delta) + \delta$  or  $\delta > \frac{1}{3}$
- ▶ Thus strategy pair  $(s_1^*, s_2^*)$  is *not* SPE if  $\delta > \frac{1}{3}$

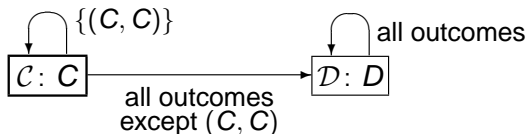
## SPE of infinitely repeated *Prisoner's Dilemma*

- ▶ Is there another strategy pair that generates the outcome path  $((C, C), (C, C), \dots)$  and is a SPE?
- ▶ Consider variant of strategy  $s_i^*$
- ▶ Grim strategy is



## SPE of infinitely repeated *Prisoner's Dilemma*

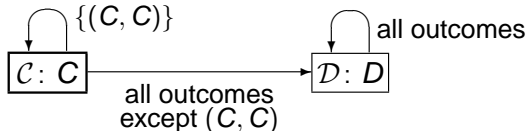
	C	D
C	3, 3	0, 4
D	4, 0	1, 1



- ▶ Consider subgame following history  $(C, D)$
- ▶ Assume that P2 uses grim strategy
- ▶ P1 uses strategy grim strategy in subgame
  - ⇒ outcome in subgame is  $(D, D)$  in every subsequent period
  - ⇒ discounted average payoff 1 to P1
- ▶ P1 uses any other strategy in subgame
  - ⇒ outcome in subgame is either  $(C, D)$  or  $(D, D)$  in every subsequent period
  - ⇒ discounted average payoff of at most 1 to P1
- ▶ Thus strategy pair in which both players use grim strategy is NE of subgame

## SPE of infinitely repeated *Prisoner's Dilemma*

	C	D
C	3, 3	0, 4
D	4, 0	1, 1



- ▶ In every subgame, either both players' strategies are in state  $C$  or both players' strategies are in state  $D$
- ▶ Both strategies in state  $C$ 
  - ⇒ (*Grim strategy*, *Grim strategy*) is NE if players are sufficiently patient (by argument in last class)
- ▶ Both strategies in state  $D$ 
  - ⇒ (*Grim strategy*, *Grim strategy*) is NE (by argument for subgame following  $(C, D)$ )
- ▶ So if players are sufficiently patient, (*Grim strategy*, *Grim strategy*) is SPE, with outcome  $(C, C)$  in every period



## SPE of infinitely repeated *Prisoner's Dilemma*

- ▶ In fact, every strictly enforceable payoff pair in *Prisoner's Dilemma* can be achieved in a SPE
- ▶ That is, set of payoffs to NEs is same as set of payoffs to SPEs
- ▶ To show result, will use equivalence of SPEs and strategy profiles satisfying one-deviation property in infinitely repeated games with discounting

### Proposition

A strategy profile is a subgame perfect equilibrium of a  $\delta$ -discounted infinitely repeated game if and only if it satisfies the one-deviation property.

(Lemma 153.1 in book)

# Subgame perfect folk theorem for infinitely repeated *Prisoner's Dilemma*

**Proposition (*Subgame perfect folk theorem for infinitely repeated Prisoner's Dilemma*)**

Let  $x$  be a strictly enforceable payoff pair in the *Prisoner's Dilemma*. For all  $\varepsilon > 0$  there exists  $\underline{\delta} < 1$  such that if  $\delta > \underline{\delta}$  then the  $\delta$ -discounted infinitely repeated game of the *Prisoner's Dilemma* has a subgame perfect equilibrium in which the discounted average payoff pair  $x'$  satisfies  $|x' - x| < \varepsilon$ .

Compare with previous result:

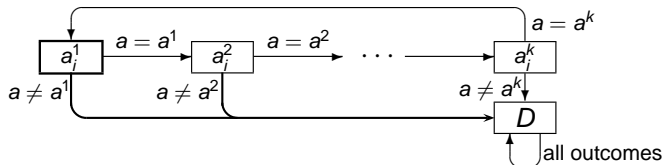
**Proposition (*Nash folk theorem*)**

Let  $x$  be a strictly enforceable payoff profile of a strategic game  $G$ . For all  $\varepsilon > 0$  there exists  $\underline{\delta} < 1$  such that if  $\delta > \underline{\delta}$  then the  $\delta$ -discounted infinitely repeated game of  $G$  has a Nash equilibrium whose payoff profile  $w'$  satisfies  $|x' - x| < \varepsilon$ .

## Proof of subgame perfect folk theorem for PD

- ▶  $(x_1, x_2)$  feasible  $\Rightarrow$  for  $\delta$  close to 1 we can find outcome path  $((a^1, a^2, \dots, a^k), (a^1, a^2, \dots, a^k), \dots)$  of repeated game for which payoff pair is close to  $(x_1, x_2)$
- ▶ Suppose player  $i$  uses following strategy:

	C	D
C	3, 3	0, 4
D	4, 0	1, 1

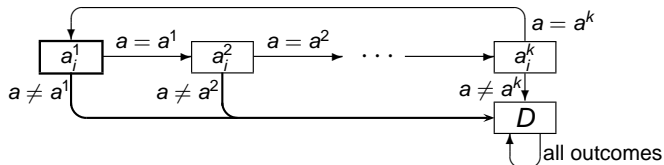


- ▶ Subgame following history in which neither player has deviated from equilibrium path:
  - ▶  $P_i$  adheres to strategy  $\Rightarrow$  payoff close to  $x_i$
  - ▶  $P_i$  deviates in first period of subgame, then follows strategy  $\Rightarrow$  payoff  $\leq 4$  in period of deviation and 1 subsequently
  - ▶ If  $\delta$  is close enough to 1, adhering to strategy is better than deviating, given that  $x_i > 1$  ( $(x_1, x_2)$  is strictly enforceable)

## Proof of subgame perfect folk theorem for PD

- ▶  $(x_1, x_2)$  feasible  $\Rightarrow$  for  $\delta$  close to 1 we can find outcome path  $((a^1, a^2, \dots, a^k), (a^1, a^2, \dots, a^k), \dots)$  of repeated game for which payoff pair is close to  $(x_1, x_2)$
- ▶ Suppose player  $i$  uses following strategy:

	C	D
C	3, 3	0, 4
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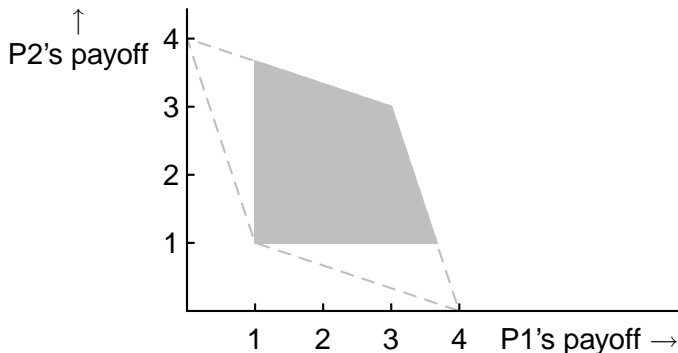


- ▶ Subgame following a history in which a player has deviated from the equilibrium path:
  - ▶ each player chooses  $D$  in every period regardless of the other player's action, so no deviation increases deviator's payoff

# SPE of infinitely repeated *Prisoner's Dilemma*

## Conclusion

Set of payoffs to subgame perfect equilibria of *Prisoner's Dilemma* is essentially same as set of payoffs to Nash equilibria: at least the set of all strictly enforceable payoffs and not more than the set of enforceable payoffs



# SPE of general repeated two-player games

- ▶ *Prisoner's Dilemma* is special: has Nash equilibrium in which each player's payoff is her minmax payoff

	C	D
C	3, 3	0, 4
D	4, 0	1, 1

- ▶ In any game, for each player

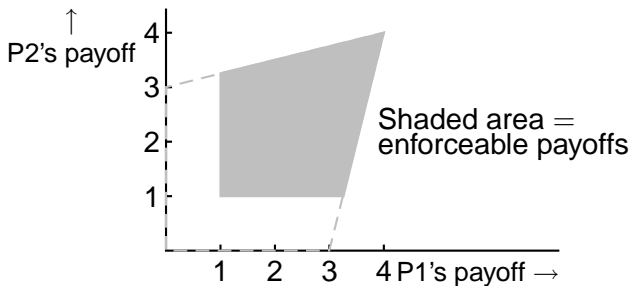
$$\text{NE payoff} \geq \text{minmax payoff}$$

but game may have no NE in which payoff = minmax payoff for each player

# SPE of general infinitely repeated two-player games

## Example

	A	B	C
A	4, 4	3, 0	1, 0
B	0, 3	2, 2	1, 0
C	0, 1	0, 1	0, 0



- ▶ Minmax payoffs:  $(1, 1)$
- ▶ Nash equilibrium:  $(A, A)$ , with payoffs  $(4, 4)$

# SPE of general infinitely repeated two-player games

	A	B	C
A	4, 4	3, 0	1, 0
B	0, 3	2, 2	1, 0
C	0, 1	0, 1	0, 0

Minmax payoffs:  $(1, 1)$

Nash equilibrium:  $(A, A)$

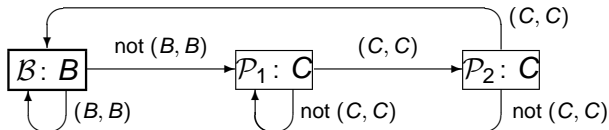
- ▶ In infinitely repeated game, can average payoffs between 1 and 4 be achieved in a SPE?
- ▶ Consider possibility of SPE that generates path in which outcome is  $(B, B)$  in every period
- ▶ Clearly cannot use Nash equilibrium,  $(A, A)$ , as punishment for deviation
- ▶ Need to make it worthwhile for a player to carry out punishment: she must be made worse off if she fails to punish



# SPE of general infinitely repeated two-player games

	A	B	C
A	4, 4	3, 0	1, 0
B	0, 3	2, 2	1, 0
C	0, 1	0, 1	0, 0

Consider strategy:

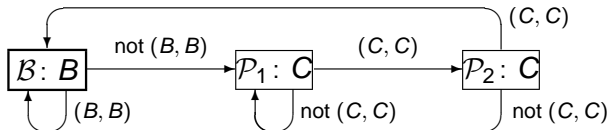


- ▶ Two-period punishment after deviation from  $(B, B)$
- ▶ If both players choose  $C$  during punishment phase then after two periods they both revert to  $B$
- ▶ If one player does not choose  $C$  in first period of punishment then punishment restarts
- ▶ Deviation from  $C$  in second period of punishment  $\Rightarrow$  transition to first punishment state: punishment restarts

# SPE of general infinitely repeated two-player games

	A	B	C
A	4, 4	3, 0	1, 0
B	0, 3	2, 2	1, 0
C	0, 1	0, 1	0, 0

Consider strategy:

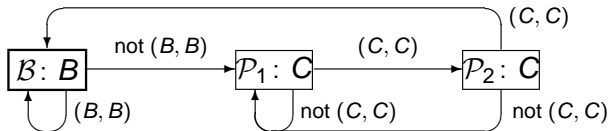


- ▶ Player is punished for not carrying out punishment
- ▶ SPE for both players to use this strategy?
- ▶ Suppose P2 adheres to strategy. Can P1 increase her payoff by deviating at the start of a subgame, holding rest of her strategy fixed?
- ▶ After any history, both players' automata are in same state, so need to consider only three cases

# SPE of general infinitely repeated two-player games

	A	B	C
A	4, 4	3, 0	1, 0
B	0, 3	2, 2	1, 0
C	0, 1	0, 1	0, 0

Consider strategy:



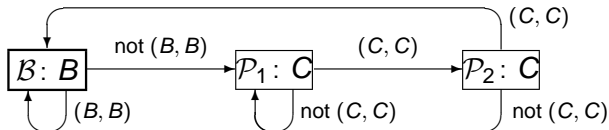
## State $B$

- ▶ P1 adheres to strategy  $\Rightarrow$  payoffs  $2, 2, 2, 2, \dots$
- ▶ P1 deviates  $\Rightarrow$  payoffs  $(3 \text{ or } 0), 0, 0, 2, 2, \dots$
- ▶ So adhering to strategy is optimal if  $2 + 2\delta + 2\delta^2 \geq 3$ , or  $\delta \geq \frac{1}{2}(\sqrt{3} - 1) \approx 0.366$

# SPE of general infinitely repeated two-player games

	A	B	C
A	4, 4	3, 0	1, 0
B	0, 3	2, 2	1, 0
C	0, 1	0, 1	0, 0

Consider strategy:



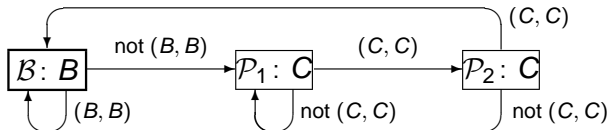
## State $\mathcal{P}_1$

- ▶ P1 adheres to strategy  $\Rightarrow$  payoffs 0, 0, 2, 2, 2, ...
- ▶ P1 deviates  $\Rightarrow$  payoffs 1, 0, 0, 2, 2, ...
- ▶ So adhering to strategy is optimal if  $2\delta^2 \geq 1$ , or  $\delta \geq \frac{1}{2}\sqrt{2} \approx 0.707$

# SPE of general infinitely repeated two-player games

	A	B	C
A	4, 4	3, 0	1, 0
B	0, 3	2, 2	1, 0
C	0, 1	0, 1	0, 0

Consider strategy:



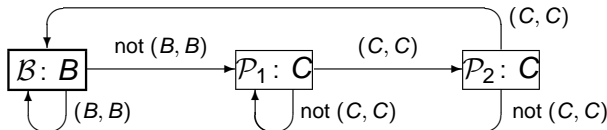
## State $\mathcal{P}_2$

- ▶ P1 adheres to strategy  $\Rightarrow$  payoffs 0, 2, 2, 2, 2, ...
- ▶ P1 deviates  $\Rightarrow$  payoffs 1, 0, 0, 2, 2, ...
- ▶ So adhering to strategy is optimal if  $2\delta + 2\delta^2 \geq 1$ , or certainly if  $2\delta^2 \geq 1$

# SPE of general infinitely repeated two-player games

	A	B	C
A	4, 4	3, 0	1, 0
B	0, 3	2, 2	1, 0
C	0, 1	0, 1	0, 0

Consider strategy:



## Conclusion

We have  $\frac{1}{2}(\sqrt{3} - 1) < \frac{1}{2}\sqrt{2}$ , so strategy pair in which both players use this strategy is subgame perfect equilibrium if  $\delta \geq \frac{1}{2}\sqrt{2} \approx 0.707$

# SPE of general infinitely repeated two-player games

Idea behind example can be extended to any two-player game

**Proposition (*Subgame perfect equilibrium folk theorem for two-player games*)**

Every strictly enforceable payoff profile of a two-player strategic game  $G$  is (at least) arbitrarily close to a subgame perfect equilibrium payoff profile of the  $\delta$ -discounted infinitely repeated game of  $G$  when  $\delta$  is sufficiently close to 1.

- ▶ Result can be extended to  $n$ -player games in which the set of feasible payoffs is  $n$ -dimensional (Proposition 151.1 in book)

## Finitely repeated games: Nash equilibrium

Consider a game played a fixed finite number of times

Example: *Prisoner's Dilemma*

	C	D
C	3, 3	0, 4
D	4, 0	1, 1

*Claim:* In every Nash equilibrium of finitely repeated *Prisoner's Dilemma* the outcome in every period is  $(D, D)$

*Argument:*

- ▶ Suppose outcome is not  $(D, D)$  in some period
- ▶ Let  $t$  be *last* period in which outcome is not  $(D, D)$  (because horizon is finite, such  $t$  exists)
- ▶ At least one player can profitably deviate from  $a^t$ —say P1
- ▶ Consider strategy of P1 that chooses profitable deviation in period  $t$  and  $D$  subsequently, regardless of history
- ▶ This strategy is profitable deviation in repeated game



## Finitely repeated games: Nash equilibrium

- ▶ Result depends on special property of *Prisoner's Dilemma*: in unique Nash equilibrium, both players' payoffs are their minmax payoffs
- ▶ For any strategic game  $G$ , outcome in last period of repeated game must be Nash equilibrium of  $G$
- ▶ *But* if  $G$  has Nash equilibrium in which some player's payoff exceeds her minmax payoff, earlier outcomes need not be Nash equilibria of  $G$ : deviant can be punished with minmax payoff

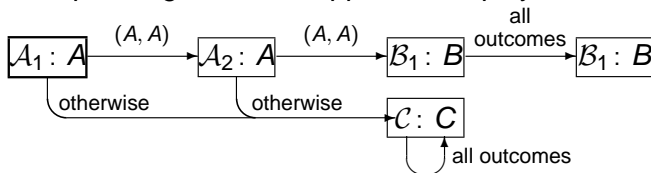
# Finitely repeated games: Nash equilibrium

## Example

	A	B	C
A	3, 3	0, 4	1, 0
B	4, 0	2, 2	1, 0
C	0, 1	0, 1	0, 0

- ▶ Unique NE:  $(B, B)$
- ▶ Minmax payoffs:  $(1, 1)$

Consider 4-period game and suppose both players use strategy



- ▶ Suppose P2 uses this strategy
- ▶ If P1 uses the strategy, outcome is  $((A, A), (A, A), (B, B), (B, B))$ , with payoffs  $(10, 10)$
- ▶ If P1 deviates in period 1, payoff is at most  $4 + 1 + 1 + 1 = 7$

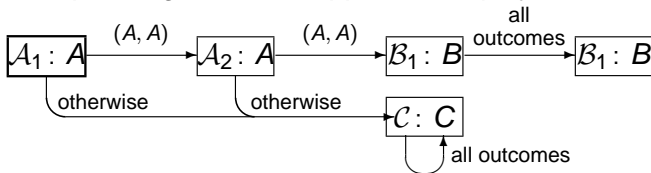
# Finitely repeated games: Nash equilibrium

## Example

	A	B	C
A	3, 3	0, 4	1, 0
B	4, 0	2, 2	1, 0
C	0, 1	0, 1	0, 0

- ▶ Unique NE:  $(B, B)$
- ▶ Minmax payoffs:  $(1, 1)$

Consider 4-period game and suppose both players use strategy



- ▶ Suppose P2 uses this strategy
- ▶ If P1 uses the strategy, outcome is  $((A, A), (A, A), (B, B), (B, B))$ , with payoffs  $(10, 10)$
- ▶ If P1 deviates in period 2, payoff is at most  $3 + 4 + 1 + 1 = 9$

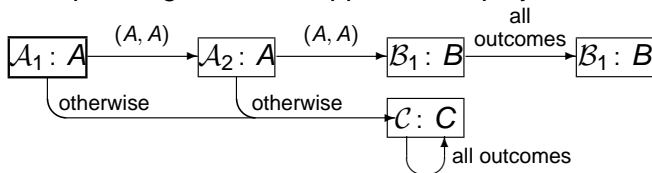
# Finitely repeated games: Nash equilibrium

## Example

	A	B	C
A	3, 3	0, 4	1, 0
B	4, 0	2, 2	1, 0
C	0, 1	0, 1	0, 0

- ▶ Unique NE:  $(B, B)$
- ▶ Minmax payoffs:  $(1, 1)$

Consider 4-period game and suppose both players use strategy



- ▶ Suppose P2 uses this strategy
- ▶ If P1 uses the strategy, outcome is  $((A, A), (A, A), (B, B), (B, B))$ , with payoffs  $(10, 10)$
- ▶ If P1 deviates in periods 3 or 4, she is worse off because  $(B, B)$  is NE of stage game

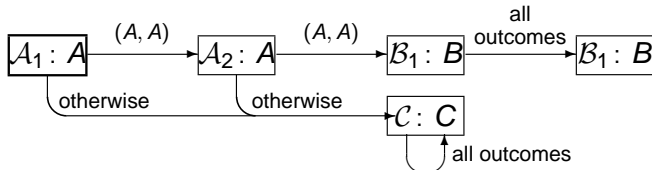
# Finitely repeated games: Nash equilibrium

## Example

	A	B	C
A	3, 3	0, 4	1, 0
B	4, 0	2, 2	1, 0
C	0, 1	0, 1	0, 0

- ▶ Unique NE:  $(B, B)$
- ▶ Minmax payoffs:  $(1, 1)$

Consider 4-period game and suppose both players use strategy



- ▶ Conclusion: strategy pair in which each player uses the strategy is a NE
- ▶ In  $T$ -period game, strategy pair in which each strategy starts with  $T - 2$  periods of  $A$  and ends with 2 periods of  $B$  is a NE

# Finitely repeated games: Nash equilibrium

## Proposition (*Nash folk theorem for finitely repeated games*)

If  $G$  has a Nash equilibrium in which the payoff of every player  $i$  exceeds her minmax payoff, then for any strictly enforceable outcome  $a^*$  of  $G$  and any  $\varepsilon > 0$  there exists  $T^*$  such that if  $T > T^*$  then the  $T$ -period repeated game of  $G$  has a Nash equilibrium in which the payoff of every player  $i$  is within  $\varepsilon$  of  $u_i(a^*)$ .

# Finitely repeated games: Nash equilibrium

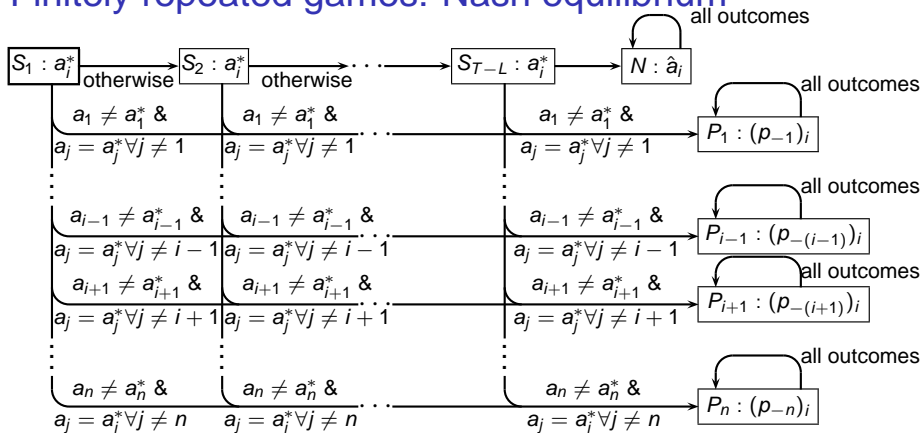
## Proof

- ▶ For each player  $j$ , let  $p_{-j}$  be a list of actions of the other players that holds  $j$ 's payoff to its minmax value,  $v_j$ :

$$p_{-j} \in \arg \min_{a_{-j} \in A_{-j}} \left( \max_{a_j \in A_j} u_j(a_{-j}, a_j) \right)$$

- ▶ Suppose each player  $i$  uses following strategy:
  - ▶ in periods  $1, \dots, T - L$  choose  $a_i^*$  until first period in which a single player  $j \neq i$  deviates, after which chooses  $(p_{-j})_i$
  - ▶ in periods  $T - L + 1, \dots, T$  choose  $i$ 's component of a Nash equilibrium  $\hat{a}$  of  $G$  in which every player's payoff exceeds her minmax payoff

# Finitely repeated games: Nash equilibrium



- ▶ Cannot profitably deviate by changing actions in last  $L$  periods because  $\hat{a}$  is NE of  $G$
- ▶ If  $L$  large enough, cannot profitably deviate by changing actions in earlier periods because  $u_i(\hat{a})$  exceeds  $i$ 's minmax payoff



# Finitely repeated games: Subgame perfect equilibrium

- ▶ In SPE, outcome in last period *after any history* must be Nash equilibrium of  $G$
- ▶ So if  $G$  has unique Nash equilibrium payoff profile, no punishment is possible

## Proposition

If  $G$  has a unique Nash equilibrium payoff profile, then for any value of  $T$  the action profile chosen after any history in any subgame perfect equilibrium of the  $T$ -period repeated game of  $G$  is a Nash equilibrium of  $G$ .

- ▶ If  $G$  has more than one Nash equilibrium payoff profile, punishment *is* possible

# Finitely repeated games: Subgame perfect equilibrium

If  $G$  has more than one Nash equilibrium payoff profile, credible punishment is possible

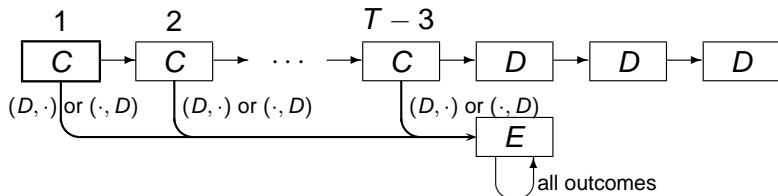
## Example

	$C$	$D$	$E$
$C$	3, 3	0, 4	0, 0
$D$	4, 0	1, 1	0, 0
$E$	0, 0	0, 0	$\frac{1}{2}, \frac{1}{2}$

Nash equilibria

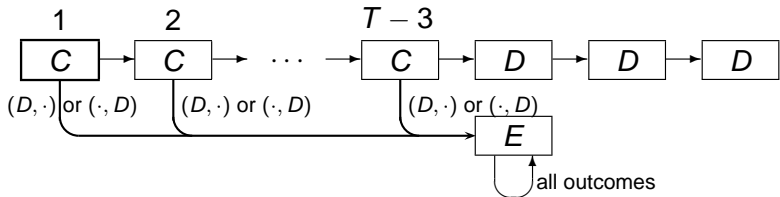
- ▶  $(D, D)$
- ▶  $(E, E)$

Strategy:



# Finitely repeated games: Subgame perfect equilibrium

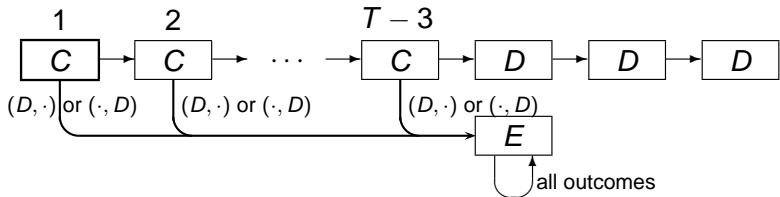
	<i>C</i>	<i>D</i>	<i>E</i>
<i>C</i>	3, 3	0, 4	0, 0
<i>D</i>	4, 0	1, 1	0, 0
<i>E</i>	0, 0	0, 0	$\frac{1}{2}, \frac{1}{2}$



- ▶ Deviation most difficult to deter: in period  $T - 3$
- ▶ Adhere to strategy  $\Rightarrow$  payoff in last 4 periods:  
 $3 + 1 + 1 + 1 = 6$
- ▶ Deviate  $\Rightarrow$  payoff in last 4 periods:  $4 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{11}{2}$

# Finitely repeated games: Subgame perfect equilibrium

	<i>C</i>	<i>D</i>	<i>E</i>
<i>C</i>	3, 3	0, 4	0, 0
<i>D</i>	4, 0	1, 1	0, 0
<i>E</i>	0, 0	0, 0	$\frac{1}{2}, \frac{1}{2}$



- ▶  $(D, D)$  and  $(E, E)$  are Nash equilibria of  $G$ , so no profitable deviation from punishment or in last 3 periods
- ▶ Hence strategy pair in which both players use this strategy is subgame perfect equilibrium
- ▶  $T$  large  $\Rightarrow$  average payoffs approach 3

# Finitely repeated games: Subgame perfect equilibrium

- ▶ This example shows how payoffs greater than the payoffs in the worst Nash equilibrium can be supported in an SPE
- ▶ In fact, any strictly enforceable payoffs can be supported
- ▶ The strategies required to do so are more complicated than the ones in the example

# Finitely repeated games: Subgame perfect equilibrium

## Proposition (*Subgame perfect Folk theorem for finitely repeated games*)

Let  $a^*$  be a strictly enforceable outcome of the two-player game  $G$ . Assume that for each  $i \in N$  there are two Nash equilibria of  $G$  that differ in the payoff of player  $i$ . Then for any  $\varepsilon > 0$  there exists an integer  $T^*$  such that if  $T > T^*$  the  $T$ -period repeated game of  $G$  has a subgame perfect equilibrium in which the payoff of each player  $i$  is within  $\varepsilon$  of  $u_i(a^*)$ .

Result can be generalized to any outcome path with strictly enforceable payoffs

As in case of infinitely repeated games, extension to many players requires restriction on dimension of set of feasible payoff profiles

# Repeated games: Summary

## Infinitely repeated games with discounting

- NE** When players are very patient, set of discounted average payoff profiles generated by Nash equilibria of repeated game is essentially set of enforceable payoff profiles of stage game
  - SPE** For two-player game, same result holds for subgame perfect equilibria
- For many player game, same result holds for subgame perfect equilibria if set of strictly enforceable outcomes has “full dimension”

# Repeated games: Summary

## Finitely repeated games

**NE** If payoff profile in every NE of stage game is profile of minmax payoffs, then set of NE outcome paths is set of sequences of NEs of the stage game

If stage game has NE in which every player's payoff exceeds her minmax payoff, then for  $T$  large enough the set of average payoff profiles generated by Nash equilibria of  $T$ -period repeated game is essentially set of enforceable payoff profiles of stage game



# Repeated games: Summary

## Finitely repeated games

**SPE** If stage game has unique NE payoff profile, then set of SPE outcome paths is set of sequences of NEs of the stage game

For two-player game, if, for each player, stage game has two NEs in which the player's payoff is different, then for  $T$  large enough the set of average payoff profiles generated by Nash equilibria of  $T$ -period repeated game is essentially set of enforceable payoff profiles of stage game

For many player game, same result holds for subgame perfect equilibria if set of strictly enforceable outcomes has "full dimension"

# Dynamic games

(Based on slides written by Colin Stewart.)

- ▶ In repeated game, payoffs in each period depend only on current actions
- ▶ But in some applications, payoffs depend directly on past actions
  - ▶ Firms' investment in capital may affect their costs over many periods
  - ▶ Firm selling durable good can keep current production as future stock
  - ▶ Individuals who extract resources from a common pool affect the quantity available in the future
- ▶ Can model this dependence by allowing payoffs to depend on a state variable

## Dynamic game: definition

A **dynamic game** (or **stochastic game**) consists of

- ▶ a set  $N$  of players
- ▶ for each player  $i$ , a set  $A_i$  of actions
- ▶ a set  $S$  of states (finite)
- ▶ for each player  $i$ , a (Bernoulli) payoff function  $u_i : A \times S \rightarrow \mathbb{R}$
- ▶ a common discount factor  $\delta \in (0, 1)$
- ▶ a probability distribution  $q_0$  over  $S$  specifying the probability of each state in the first period
- ▶ for each  $a \in A$  and  $s, s' \in S$ , a probability  $q(s' | s, a)$  that state  $s'$  occurs in period  $t$  if, in period  $t - 1$ , the state is  $s$  and the action profile is  $a$

Total payoff for player  $i$  given states  $(s^1, s^2, \dots)$  and action profiles  $(a^1, a^2, \dots)$ :

$$\sum_{t=1}^{\infty} \delta^{t-1} u_i(a^t, s^t)$$

# Special cases

- ▶ Infinitely repeated games
  - ▶ Payoffs independent of state
    - ▶ Example: One state ( $S = \{s\}$ ) and transition probabilities  $q(s | s, a) = 1$  for all  $a$
- ▶ Finitely repeated games (with discounting)
  - ▶ For  $T$ -period repetition of game  $G$ , state space  $S = \{s_1, \dots, s_T, s_{T+1}\}$
  - ▶ Initial distribution  $q_0(s_1) = 1$
  - ▶ Transition probabilities  $q(s_t | s_{t-1}, a) = 1$  for all  $a$  and all  $t = 2, \dots, T$ , and  $q(s_{T+1} | s_{T+1}, a) = 1$  for all  $a$
  - ▶ Payoffs  $u_i(a, s)$  same as in  $G$  in states  $s_1, \dots, s_T$ , all payoffs equal to 0 in state  $s_{T+1}$
- ▶ Game with random payoffs
  - ▶ Transition probabilities  $q(s' | s, a)$  independent of  $a$

## Subgame perfect equilibrium

Every dynamic game corresponds to an extensive game with simultaneous and chance moves

- ▶ First chance moves according to  $q_0$
- ▶ Players move simultaneously in each period
- ▶ At the end of each period, chance moves to determine the state in the following period according to  $q$
- ▶ Perfect information: all players observe each move by chance

A (pure) strategy for player  $i$  is a function

$\sigma_i : \bigcup_{t=1}^{\infty} (A^{t-1} \times S^t) \rightarrow A_i$  specifying an action for each history  $(a^1, \dots, a^{t-1}, s^1, \dots, s^t)$

A strategy profile is a **subgame perfect equilibrium** of a dynamic game if it is a subgame perfect equilibrium of the corresponding extensive game

## Markov perfect equilibrium

Researchers usually focus on strategies specifying actions that depend only on current state, not directly on history of actions and previous states

- ▶ A strategy  $\sigma_i$  for player  $i$  is **Markov** if there exists a function  $\tilde{\sigma}_i : S \rightarrow A_i$  such that  $\sigma_i(a^1, \dots, a^{t-1}, s^1, \dots, s^t) = \tilde{\sigma}_i(s^t)$  for all  $(a^1, \dots, a^{t-1}, s^1, \dots, s^t)$

Note: some authors allow Markov strategies to depend also on time, and refer to strategies of the above form as “stationary Markov”

A strategy profile of a dynamic game is a **Markov perfect equilibrium (MPE)** if it is a subgame perfect equilibrium and each player's strategy is Markov

Can show that MPE in possibly mixed strategies exists if  $A$  and  $S$  are finite

# Example 1

Consider infinitely repeated Prisoner's dilemma

	<i>C</i>	<i>D</i>
<i>C</i>	3, 3	0, 4
<i>D</i>	4, 0	1, 1

Many SPEs if  $\delta$  is close to 1 (Folk Theorem)

Which SPEs are MPEs?

- ▶ Suppose there is a single state  $s$
- ▶ Strategy is Markov if and only if it chooses same action *regardless of history*
- ▶ Not SPE to choose  $C$  at every history: no punishment, so each player prefers to deviate
- ▶  $\Rightarrow$  unique MPE: each player always chooses  $D$

## MPE in repeated games

MPE does not allow players to respond to opponents' past behavior unless it affects current state

- ▶ In repeated game of  $G$ , MPE for dynamic game with single state consists of repeated play of a NE of  $G$
- ▶ But in many contexts, the idea that one might punish an opponent for their past behavior (even if that behavior does not directly affect current payoffs) seems reasonable
- ▶ Suggests that one should be cautious about whether MPE is an appropriate concept even outside of repeated games
- ▶ Typical justification of MPE is based on analytical convenience



## Example 2

Two states, equally likely in each period *regardless of history*

	C	D
C	3, 3	0, 4
D	4, 0	1, 1

	C	D
C	5, 5	0, 6
D	6, 0	1, 1

Does game have SPE with outcome (C, C) in every period?

- ▶ Consider pair of grim strategies: after any deviation, play  $D$  forever in both states
- ▶ After any history with no deviations, payoff from equilibrium strategy in state 1:

$$3 + \sum_{t=1}^{\infty} \delta^t \left( \frac{1}{2} \cdot 3 + \frac{1}{2} \cdot 5 \right) = 3 + \frac{4\delta}{1-\delta}$$

- ▶ Payoff from deviating to  $D$  in state 1:  $4 + \frac{\delta}{1-\delta}$
- ▶ Prefer not to deviate if  $\delta \geq \frac{1}{4}$

## Example 2

Two states, equally likely in each period *regardless of history*

	C	D
C	3, 3	0, 4
D	4, 0	1, 1

	C	D
C	5, 5	0, 6
D	6, 0	1, 1

Does game have SPE with outcome (C, C) in every period?

- ▶ Consider pair of grim strategies: after any deviation, play *D* forever in both states
- ▶ After any history with no deviations, payoff from equilibrium strategy in state 2:

$$5 + \sum_{t=1}^{\infty} \delta^t \left( \frac{1}{2} \cdot 3 + \frac{1}{2} \cdot 5 \right) = 5 + \frac{4\delta}{1-\delta}$$

- ▶ Payoff from deviating to *D* in state 1:  $6 + \frac{\delta}{1-\delta}$
- ▶ Prefer not to deviate if  $\delta \geq \frac{1}{4}$

## Example 2

Two states, equally likely in each period *regardless of history*

	C	D
C	3, 3	0, 4
D	4, 0	1, 1

	C	D
C	5, 5	0, 6
D	6, 0	1, 1

Does game have SPE with outcome (C, C) in every period?

- ▶ Consider pair of grim strategies: after any deviation, play  $D$  forever in both states
- ▶ At any history with at least one deviation, no incentive to deviate in either state

So game has SPE with outcome (C, C) in every period whenever  $\delta \geq \frac{1}{4}$

## Example 2

Two states, equally likely in each period regardless of history

	C	D
C	3, 3	0, 4
D	4, 0	1, 1

	C	D
C	5, 5	0, 6
D	6, 0	1, 1

Does game have SPE in which outcome is (C, C) in state 1, (D, D) in state 2?

- ▶ Consider pair of strategies: in state 1, choose C then D after any deviation, and in state 2, choose D always
- ▶ After any history with no deviations, payoff from equilibrium strategy in state 1:

$$3 + \sum_{t=1}^{\infty} \delta^t \left( \frac{1}{2} \cdot 3 + \frac{1}{2} \cdot 1 \right) = 3 + \frac{2\delta}{1-\delta}$$

- ▶ Payoff from deviating to D in state 1 equals  $4 + \frac{\delta}{1-\delta}$
- ▶ Prefer not to deviate if  $\delta \geq \frac{1}{2}$

## Example 2

Two states, equally likely in each period regardless of history

	C	D
C	3, 3	0, 4
D	4, 0	1, 1

	C	D
C	5, 5	0, 6
D	6, 0	1, 1

Does game have SPE in which outcome is  $(C, C)$  in state 1,  $(D, D)$  in state 2?

- ▶ Consider pair of strategies: in state 1, choose  $C$  then  $D$  after any deviation, and in state 2, choose  $D$  always
- ▶ No incentive to deviate in state 2, or in state 1 at any history with at least one deviation

$\Rightarrow$  game has such an SPE iff  $\delta \geq \frac{1}{2}$

Failure to cooperate in one state makes it more difficult to sustain cooperation in the other one

## Example 2

Two states, equally likely in each period regardless of history

	<i>C</i>	<i>D</i>
<i>C</i>	3, 3	0, 4
<i>D</i>	4, 0	1, 1

	<i>C</i>	<i>D</i>
<i>C</i>	5, 5	0, 6
<i>D</i>	6, 0	1, 1

What are the MPEs of this game?

- ▶ Markov strategy specifies pair of actions, one for each state
- ▶ If each player uses Markov strategy, deviating in one period has no effect on future play

⇒ unique MPE involves both players choosing *D* in both states

## Example 3

Two states, *identical* payoffs

	<i>C</i>	<i>D</i>
<i>C</i>	3, 3	0, 4
<i>D</i>	4, 0	1, 1

	<i>C</i>	<i>D</i>
<i>C</i>	3, 3	0, 4
<i>D</i>	4, 0	1, 1

Begin in state 1; remain in state 1 as long as both choose *C*;  
switch to state 2 forever if either player chooses *D* in any period

What are the MPE of this game?

- ▶ Markov strategy consists of a pair of actions: one to be chosen in state 1, the other in state 2
- ▶ Every subgame in state 2 is identical to repeated PD  $\Rightarrow$  both players must choose *D* in MPE
- ▶ It one MPE, both choose *D* also in state 1
- ▶ But if both choose *C* in state 1, then strategies are equivalent to grim strategies  $\Rightarrow$  MPE if  $\delta \geq \frac{1}{3}$

## Example 3: Discussion

With one state, game has a unique MPE; with two identical states, game has two MPE, including one in which players do not play the NE of stage game

- ▶ Raises question of how to define state space
- ▶ If state is payoff-irrelevant, should we allow behavior to depend on it?
- ▶ Some argue that states should be defined according to payoff-relevance: no two states should be identical in terms of payoffs
- ▶ But this assumption is not entirely satisfactory: with only a very small difference between payoffs in the two states, the example goes through (e.g., there could be an additional tiny benefit associated with the first defection)



# Summary

- ▶ Dynamic games often have a large set of SPE
- ▶ The set of MPE is typically much smaller
- ▶ But MPE may not be appropriate in settings where players might punish deviators
- ▶ Set of MPEs depend on the state space