ECO2030: Microeconomic Theory II, module 1

Lecture 11

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Nash equilibrium of infinitely repeated games

- When players are sufficiently patient, set of Nash equilibrium payoff profiles of infinitely repeated game with discounting is approximately equal to set of strictly enforceable payoff profiles of stage game

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Nash equilibrium of infinitely repeated games

- When players are sufficiently patient, set of Nash equilibrium payoff profiles of infinitely repeated game with discounting is approximately equal to set of strictly enforceable payoff profiles of stage game
- Equilibrium strategies involve “punishments” for players who deviate from norm
- Are the Nash equilibria subgame perfect?

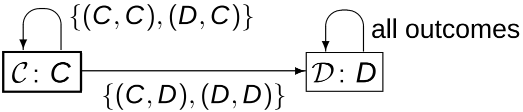
SPE of infinitely repeated *Prisoner's Dilemma*

Consider Nash equilibrium of infinitely repeated *Prisoner's Dilemma*

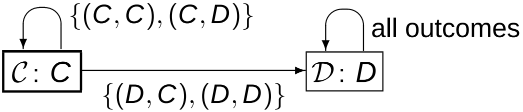
	C	D
C	$3, 3$	$0, 4$
D	$4, 0$	$1, 1$

in which players' strategies are

s_1^*:

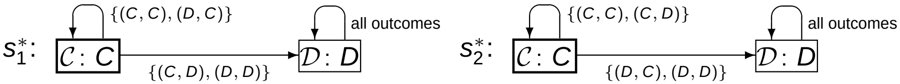


s_2^*:

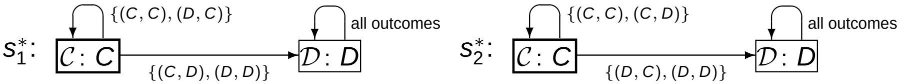


Is this strategy pair a SPE?

SPE of infinitely repeated *Prisoner's Dilemma*

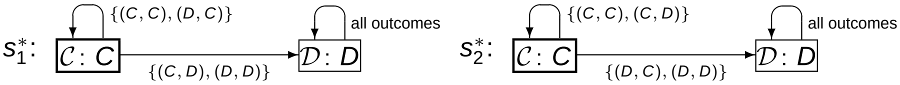


SPE of infinitely repeated *Prisoner's Dilemma*



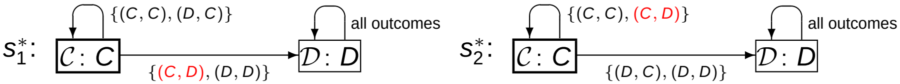
- Consider subgame following history (C, D)

SPE of infinitely repeated *Prisoner's Dilemma*



- Consider subgame following history (C, D)
- Suppose P2 uses s_2^*

SPE of infinitely repeated *Prisoner's Dilemma*



- Consider subgame following history (C, D)
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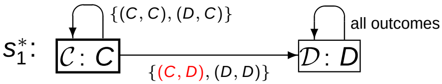
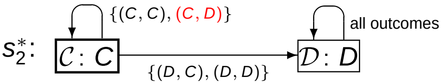
SPE of infinitely repeated *Prisoner's Dilemma*

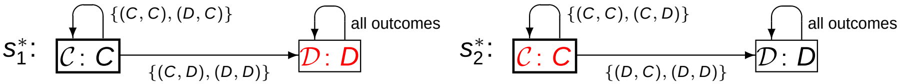




- Consider subgame following history (C, D)
- Suppose P2 uses s_2^*
- P1 uses s_1^* in subgame

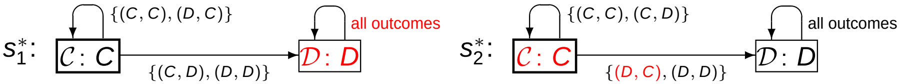
  $\Rightarrow$ outcome path in subgame is

SPE of infinitely repeated *Prisoner's Dilemma*



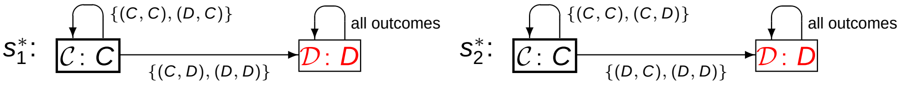
- Consider subgame following history (C, D)
- Suppose P2 uses s_2^*
- P1 uses s_1^* in subgame
  - $\Rightarrow$ outcome path in subgame is $((D, C),$

SPE of infinitely repeated *Prisoner's Dilemma*



- Consider subgame following history (C, D)
- Suppose P2 uses s_2^*
- P1 uses s_1^* in subgame
  - $\Rightarrow$ outcome path in subgame is $((D, C),$

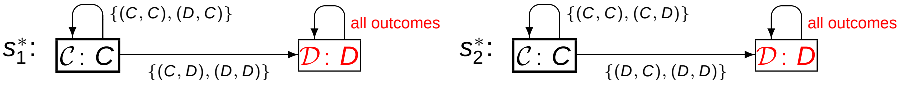
SPE of infinitely repeated *Prisoner's Dilemma*



- Consider subgame following history (C, D)
- Suppose P2 uses s_2^*
- P1 uses s_1^* in subgame

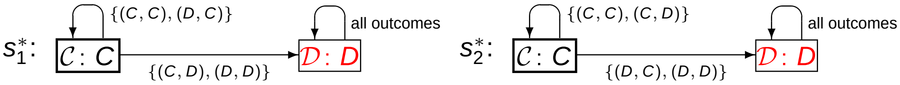
  $\Rightarrow$ outcome path in subgame is $((D, C), (D, D),$

SPE of infinitely repeated *Prisoner's Dilemma*



- Consider subgame following history (C, D)
- Suppose P2 uses s_2^*
- P1 uses s_1^* in subgame
  - $\Rightarrow$ outcome path in subgame is $((D, C), (D, D),$

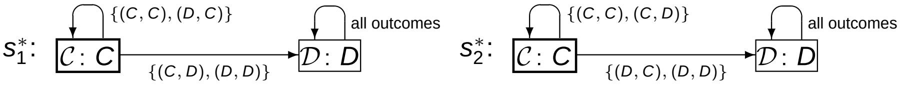
SPE of infinitely repeated *Prisoner's Dilemma*



- Consider subgame following history (C, D)
- Suppose P2 uses s_2^*
- P1 uses s_1^* in subgame

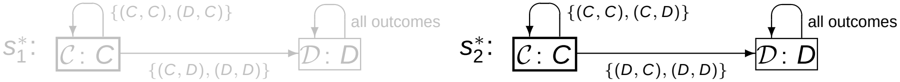
  $\Rightarrow$ outcome path in subgame is $((D, C), (D, D), (D, D), \ldots)$

SPE of infinitely repeated *Prisoner's Dilemma*



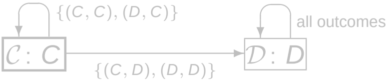
- Consider subgame following history (C, D)
- Suppose P2 uses s_2^*
- P1 uses s_1^* in subgame
  - $\Rightarrow$ outcome path in subgame is $((D,C),(D,D),(D,D),\ldots)$
  - $\Rightarrow$ payoff stream in subgame is $4, 1, 1, \ldots$ to P1, with discounted average $4(1-\delta)+\delta$

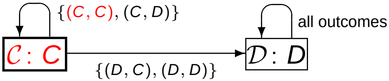
SPE of infinitely repeated *Prisoner's Dilemma*



- Consider subgame following history (C, D)
- Suppose P2 uses s_2^*
- P1 uses s_1^* in subgame
  - $\Rightarrow$ outcome path in subgame is $((D,C),(D,D),(D,D),\ldots)$
  - $\Rightarrow$ payoff stream in subgame is $4, 1, 1, \ldots$ to P1, with discounted average $4(1-\delta)+\delta$
- P1 deviates in subgame to strategy that chooses C regardless of history

SPE of infinitely repeated *Prisoner's Dilemma*

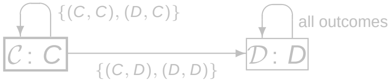
s_1^*:


s_2^*:


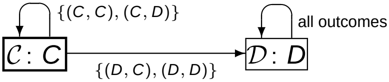
- Consider subgame following history (C, D)
- Suppose P2 uses s_2^*
- P1 uses s_1^* in subgame
  - $\Rightarrow$ outcome path in subgame is $((D, C), (D, D), (D, D), \ldots)$
  - $\Rightarrow$ payoff stream in subgame is $4, 1, 1, \ldots$ to P1, with discounted average $4(1 - \delta) + \delta$
- P1 deviates in subgame to strategy that chooses C regardless of history
  - $\Rightarrow$ outcome in subgame is (C, C) in every period

SPE of infinitely repeated *Prisoner's Dilemma*

s_1^*:

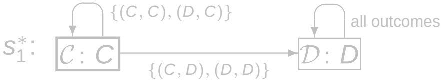


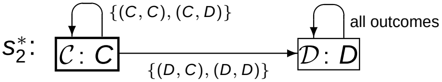
s_2^*:



- Consider subgame following history (C, D)
- Suppose P2 uses s_2^*
- P1 uses s_1^* in subgame
  - $\Rightarrow$ outcome path in subgame is $((D, C), (D, D), (D, D), \ldots)$
  - $\Rightarrow$ payoff stream in subgame is $4, 1, 1, \ldots$ to P1, with discounted average $4(1-\delta)+\delta$
- P1 deviates in subgame to strategy that chooses C regardless of history
  - $\Rightarrow$ outcome in subgame is (C, C) in every period
  - $\Rightarrow$ discounted average payoff 3 to P1

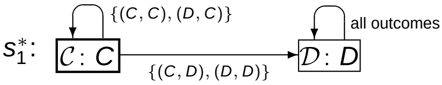
SPE of infinitely repeated *Prisoner's Dilemma*

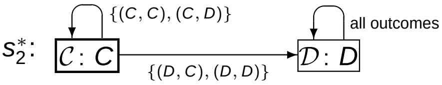




- Consider subgame following history (C, D)
- Suppose P2 uses s_2^*
- P1 uses s_1^* in subgame
  - $\Rightarrow$ outcome path in subgame is $((D, C), (D, D), (D, D), \ldots)$
  - $\Rightarrow$ payoff stream in subgame is $4, 1, 1, \ldots$ to P1, with discounted average $4(1 - \delta) + \delta$
- P1 deviates in subgame to strategy that chooses C regardless of history
  - $\Rightarrow$ outcome in subgame is (C, C) in every period
  - $\Rightarrow$ discounted average payoff 3 to P1
- Better for P1 to deviate if $3 > 4(1 - \delta) + \delta$ or $\delta > \frac{1}{3}$

SPE of infinitely repeated *Prisoner's Dilemma*





- Consider subgame following history (C, D)
- Suppose P2 uses s_2^*
- P1 uses s_1^* in subgame
  - $\Rightarrow$ outcome path in subgame is $((D,C),(D,D),(D,D),\ldots)$
  - $\Rightarrow$ payoff stream in subgame is $4, 1, 1, \ldots$ to P1, with discounted average $4(1-\delta)+\delta$
- P1 deviates in subgame to strategy that chooses C regardless of history
  - $\Rightarrow$ outcome in subgame is (C, C) in every period
  - $\Rightarrow$ discounted average payoff 3 to P1
- Better for P1 to deviate if $3 > 4(1-\delta)+\delta$ or $\delta > \frac{1}{3}$
- Thus strategy pair (s_1^*, s_2^*) is *not* SPE if $\delta > \frac{1}{3}$

SPE of infinitely repeated *Prisoner's Dilemma*

- Is there another strategy pair that generates the outcome path $((C, C), (C, C), \ldots)$ and *is* a SPE?

SPE of infinitely repeated *Prisoner's Dilemma*

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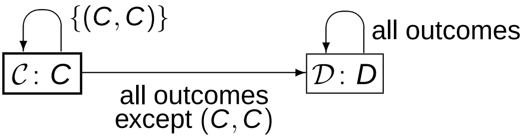
- Consider variant of strategy s_i^*

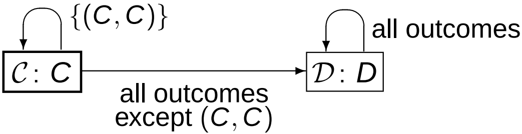
SPE of infinitely repeated *Prisoner's Dilemma*

- Is there another strategy pair that generates the outcome path $((C, C), (C, C), \ldots)$ and *is* a SPE?
- Consider variant of strategy s_i^*
- Grim strategy is



SPE of infinitely repeated *Prisoner's Dilemma*

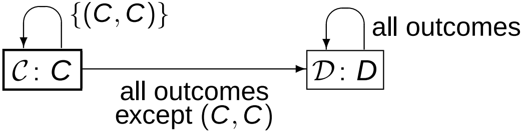
- Is there another strategy pair that generates the outcome path $((C, C), (C, C), \ldots)$ and *is* a SPE?
- Consider variant of strategy s_i^*
- Grim strategy is



strategy switches to D after any history in which *either* player deviated from (C, C)

SPE of infinitely repeated *Prisoner's Dilemma*

	C	D
C	$3, 3$	$0, 4$
D	$4, 0$	$1, 1$



SPE of infinitely repeated *Prisoner's Dilemma*

	C	D
C	$3,3$	$0,4$
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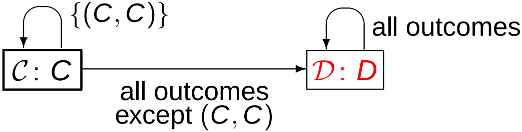

- Consider subgame following history (C,D)

SPE of infinitely repeated *Prisoner's Dilemma*

	C	D
C	$3,3$	$0,4$
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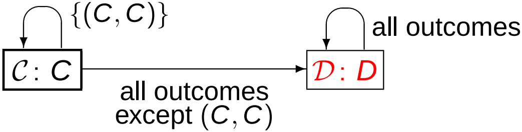

- Consider subgame following history (C,D)
- Assume that P2 uses grim strategy

SPE of infinitely repeated *Prisoner's Dilemma*

	C	D
C	$3, 3$	$0, 4$
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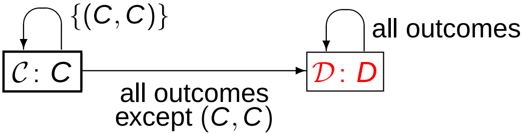

- Consider subgame following history (C, D)
- Assume that P2 uses grim strategy
- P1 uses strategy grim strategy in subgame

  $\Rightarrow$ outcome in subgame is

SPE of infinitely repeated *Prisoner's Dilemma*

	C	D
C	$3,3$	$0,4$
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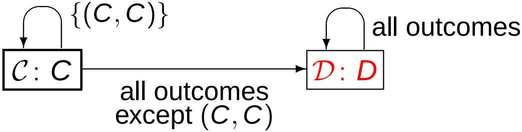
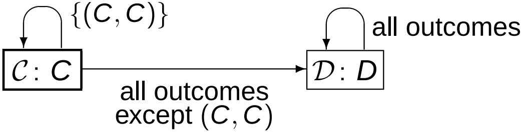


- Consider subgame following history (C, D)
- Assume that P2 uses grim strategy
- P1 uses strategy grim strategy in subgame
  - $\Rightarrow$ outcome in subgame is (D, D) in every subsequent period

SPE of infinitely repeated *Prisoner's Dilemma*

	C	D
C	$3,3$	$0,4$
D	$4,0$	$1,1$



- Consider subgame following history (C,D)
- Assume that P2 uses grim strategy
- P1 uses strategy grim strategy in subgame
  - $\Rightarrow$ outcome in subgame is (D,D) in every subsequent period
  - $\Rightarrow$ discounted average payoff 1 to P1

SPE of infinitely repeated *Prisoner's Dilemma*

	C	D
C	$3,3$	$0,4$
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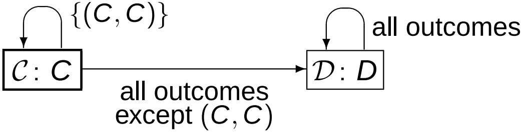

- Consider subgame following history (C,D)
- Assume that P2 uses grim strategy
- P1 uses strategy grim strategy in subgame
  - $\Rightarrow$ outcome in subgame is (D,D) in every subsequent period
  - $\Rightarrow$ discounted average payoff 1 to P1
- P1 uses any other strategy in subgame
  - $\Rightarrow$ outcome in subgame is

SPE of infinitely repeated *Prisoner's Dilemma*

	C	D
C	$3, 3$	$0, 4$
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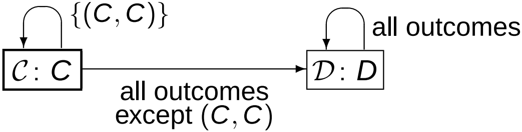

- Consider subgame following history (C, D)
- Assume that P2 uses grim strategy
- P1 uses strategy grim strategy in subgame
  - $\Rightarrow$ outcome in subgame is (D, D) in every subsequent period
  - $\Rightarrow$ discounted average payoff 1 to P1
- P1 uses any other strategy in subgame
  - $\Rightarrow$ outcome in subgame is either (C, D) or (D, D) in every subsequent period

SPE of infinitely repeated *Prisoner's Dilemma*

	C	D
C	$3,3$	$0,4$
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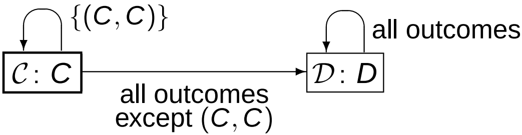

- Consider subgame following history (C,D)
- Assume that P2 uses grim strategy
- P1 uses strategy grim strategy in subgame
  - $\Rightarrow$ outcome in subgame is (D,D) in every subsequent period
  - $\Rightarrow$ discounted average payoff 1 to P1
- P1 uses any other strategy in subgame
  - $\Rightarrow$ outcome in subgame is either (C,D) or (D,D) in every subsequent period
  - $\Rightarrow$ discounted average payoff of at most 1 to P1
- Thus strategy pair in which both players use grim strategy is NE of subgame

SPE of infinitely repeated *Prisoner's Dilemma*

	C	D
C	$3, 3$	$0, 4$
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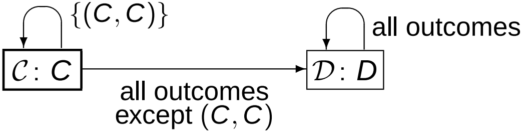

- In every subgame, either both players' strategies are in state $\mathcal{C}$ or both players' strategies are in state $\mathcal{D}$

SPE of infinitely repeated *Prisoner's Dilemma*

	C	D
C	$3, 3$	$0, 4$
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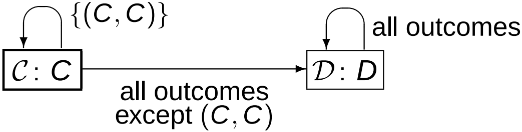

- In every subgame, either both players' strategies are in state $\mathcal{C}$ or both players' strategies are in state $\mathcal{D}$
- Both strategies in state $\mathcal{C}$

SPE of infinitely repeated *Prisoner's Dilemma*

	C	D
C	$3,3$	$0,4$
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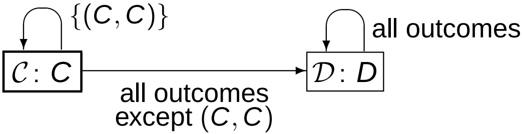
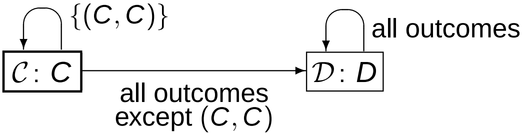


- In every subgame, either both players' strategies are in state $\mathcal{C}$ or both players' strategies are in state $\mathcal{D}$
- Both strategies in state $\mathcal{C}$
  - $\Rightarrow$ (*Grim strategy*, *Grim strategy*) is NE if players are sufficiently patient (by argument in last class)

SPE of infinitely repeated *Prisoner's Dilemma*

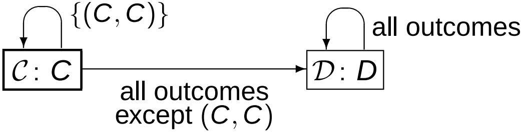
	C	D
C	$3, 3$	$0, 4$
D	$4, 0$	$1, 1$



- In every subgame, either both players' strategies are in state $\mathcal{C}$ or both players' strategies are in state $\mathcal{D}$
- Both strategies in state $\mathcal{C}$
  - $\Rightarrow$ $(\textit{Grim strategy}, \textit{Grim strategy})$ is NE if players are sufficiently patient (by argument in last class)
- Both strategies in state $\mathcal{D}$

SPE of infinitely repeated *Prisoner's Dilemma*

	C	D
C	$3, 3$	$0, 4$
D	$4, 0$	$1, 1$



- In every subgame, either both players' strategies are in state $\mathcal{C}$ or both players' strategies are in state $\mathcal{D}$
- Both strategies in state $\mathcal{C}$
  - $\Rightarrow$ (*Grim strategy*, *Grim strategy*) is NE if players are sufficiently patient (by argument in last class)
- Both strategies in state $\mathcal{D}$
  - $\Rightarrow$ (*Grim strategy*, *Grim strategy*) is NE (by argument for subgame following (C, D))

SPE of infinitely repeated *Prisoner's Dilemma*

	C	D
C	$3,3$	$0,4$
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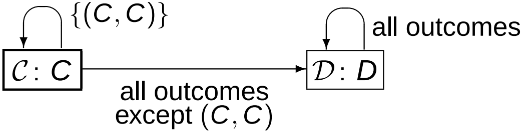

- In every subgame, either both players' strategies are in state $\mathcal{C}$ or both players' strategies are in state $\mathcal{D}$
- Both strategies in state $\mathcal{C}$
  - $\Rightarrow$ (*Grim strategy*, *Grim strategy*) is NE if players are sufficiently patient (by argument in last class)
- Both strategies in state $\mathcal{D}$
  - $\Rightarrow$ (*Grim strategy*, *Grim strategy*) is NE (by argument for subgame following (C,D))
- So if players are sufficiently patient, (*Grim strategy*, *Grim strategy*) is SPE, with outcome (C,C) in every period

SPE of infinitely repeated *Prisoner's Dilemma*

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Proposition

A strategy profile is a subgame perfect equilibrium of a δ-discounted infinitely repeated game if and only if it satisfies the one-deviation property.

(Lemma 153.1 in book)

Subgame perfect folk theorem for infinitely repeated *Prisoner's Dilemma*

Proposition (*Subgame perfect folk theorem for infinitely repeated Prisoner's Dilemma*)

Let x be a strictly enforceable payoff pair in the *Prisoner's Dilemma*. For all $\varepsilon > 0$ there exists $\underline{\delta} < 1$ such that if $\delta > \underline{\delta}$ then the δ-discounted infinitely repeated game of the *Prisoner's Dilemma* has a subgame perfect equilibrium in which the discounted average payoff pair x' satisfies $|x' - x| < \varepsilon$.

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Compare with previous result:

Proposition (*Nash folk theorem*)

Let x be a strictly enforceable payoff profile of a strategic game G. For all $\varepsilon > 0$ there exists $\underline{\delta} < 1$ such that if $\delta > \underline{\delta}$ then the δ-discounted infinitely repeated game of G has a Nash equilibrium whose payoff profile w' satisfies $|x' - x| < \varepsilon$.

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Proof of subgame perfect folk theorem for PD

- (x_1, x_2) feasible $\Rightarrow$ for δ close to 1 we can find outcome path $((a^1, a^2, \ldots, a^k), (a^1, a^2, \ldots, a^k), \ldots)$ of repeated game for which payoff pair is close to (x_1, x_2)

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	C	D
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a_i^1

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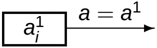

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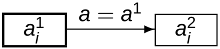

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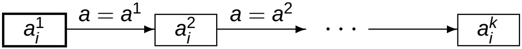

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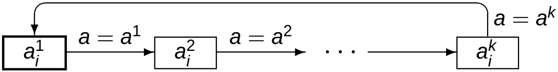

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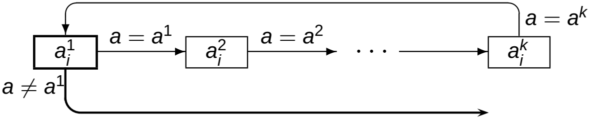

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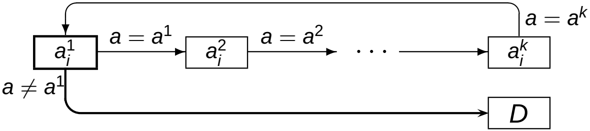

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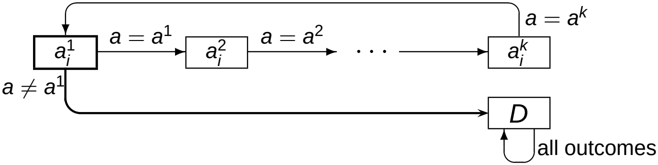

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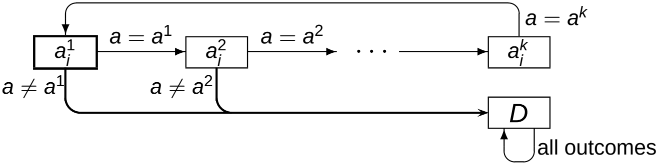

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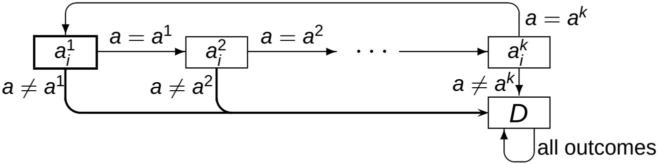

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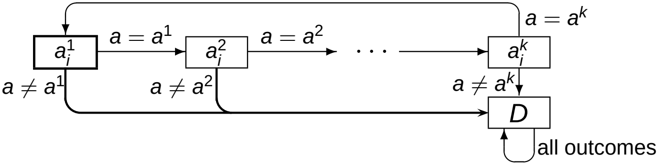

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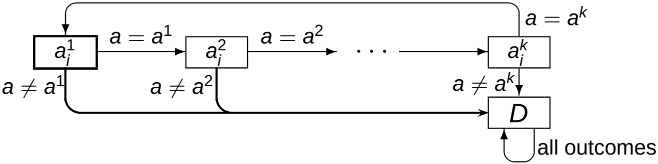

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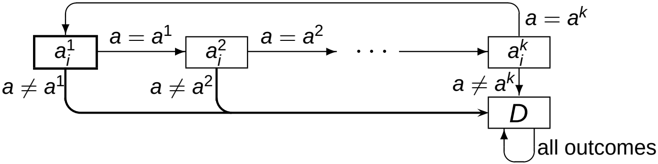

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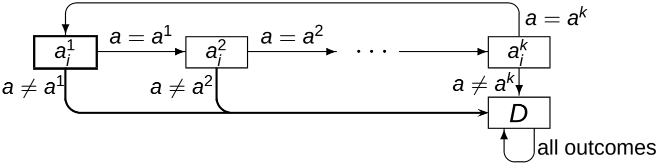

- Subgame following history in which neither player has deviated from equilibrium path:
  - Pi adheres to strategy $\Rightarrow$ payoff close to x_i
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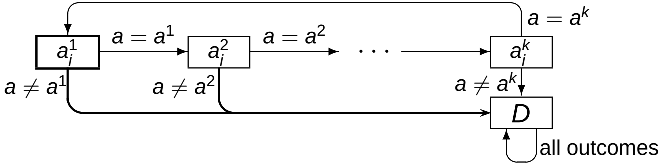

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  - Pi deviates in first period of subgame, then follows strategy $\Rightarrow$ payoff ≤ 4 in period of deviation and 1 subsequently
  - If δ is close enough to 1, adhering to strategy is better than deviating, given that $x_i > 1$ ((x_1, x_2) is strictly enforceable)

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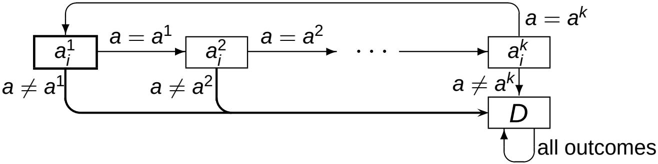

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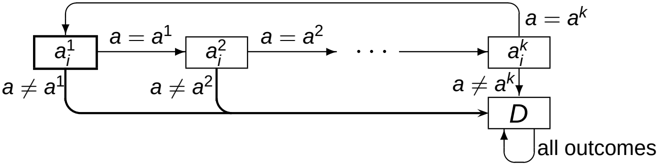

- Subgame following a history in which a player has deviated from the equilibrium path:
  - each player chooses D in every period regardless of the other player's action, so no deviation increases deviator's payoff

SPE of infinitely repeated *Prisoner's Dilemma*

Conclusion

Set of payoffs to subgame perfect equilibria of *Prisoner's Dilemma* is essentially same as set of payoffs to Nash equilibria: at least the set of all strictly enforceable payoffs and not more than the set of enforceable payoffs

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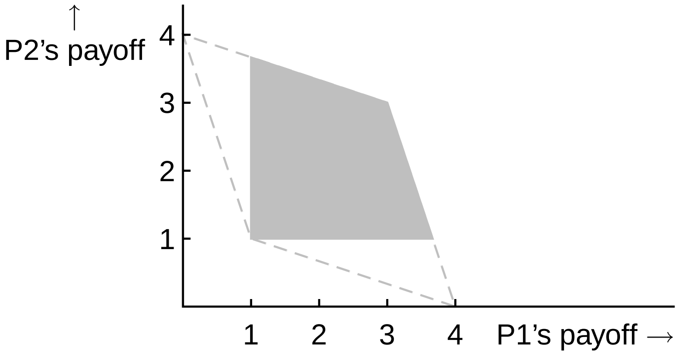

SPE of general repeated two-player games

- *Prisoner's Dilemma* is special: has Nash equilibrium in which each player's payoff is her minmax payoff

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	C	*D*
C	$3, 3$	$0, 4$
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- In any game, for each player

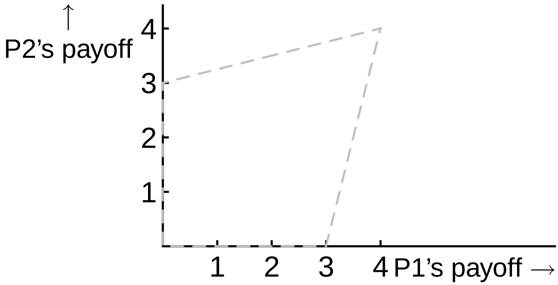
$$\text{NE payoff} \geq \text{minmax payoff}$$

but game may have no NE in which payoff $=$ minmax payoff for each player

SPE of general infinitely repeated two-player games

Example

	A	*B*	*C*
A	4, 4	3, 0	1, 0
B	0, 3	2, 2	1, 0
C	0, 1	0, 1	0, 0



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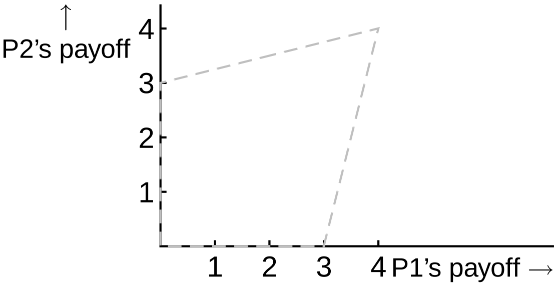

- Minmax payoffs:

SPE of general infinitely repeated two-player games

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	A	B	C
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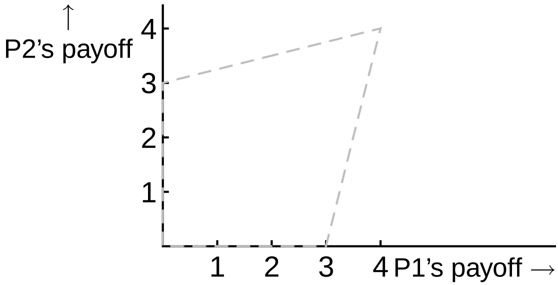

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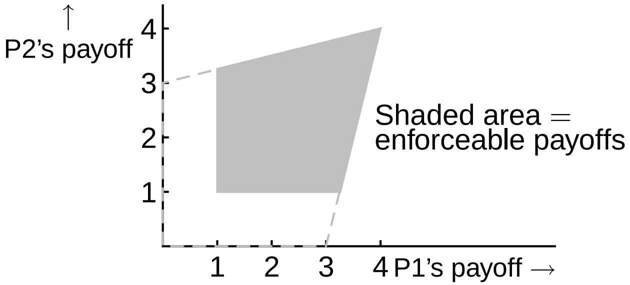

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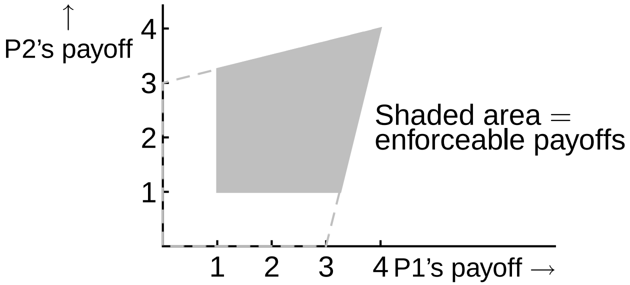

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SPE of general infinitely repeated two-player games

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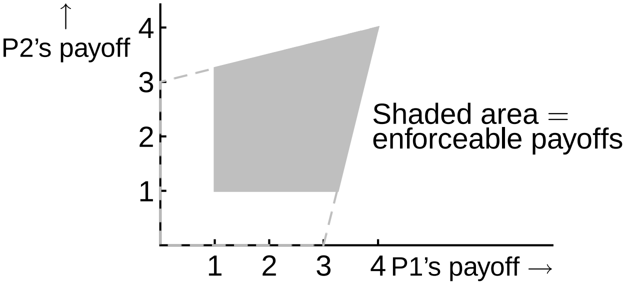

- Minmax payoffs: $(1,1)$
- Nash equilibrium: (A,A), with payoffs $(4,4)$

SPE of general infinitely repeated two-player games

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- In infinitely repeated game, can average payoffs between 1 and 4 be achieved in a SPE?

SPE of general infinitely repeated two-player games

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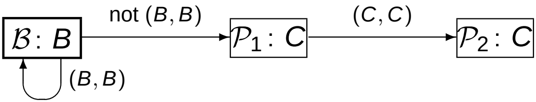
- Consider possibility of SPE that generates path in which outcome is (B, B) in every period
- Clearly cannot use Nash equilibrium, (A, A), as punishment for deviation
- Need to make it worthwhile for a player to carry out punishment: she must be made worse off if she fails to punish

SPE of general infinitely repeated two-player games

	A	B	C
A	$4, 4$	$3, 0$	$1, 0$
B	$0, 3$	$2, 2$	$1, 0$
C	$0, 1$	$0, 1$	$0, 0$

Consider strategy:



- Two-period punishment after deviation from (B, B)

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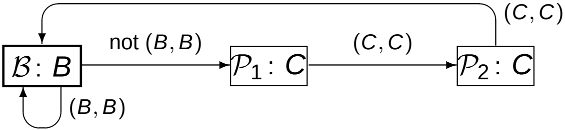

- Two-period punishment after deviation from (B,B)
- If both players choose C during punishment phase then after two periods they both revert to B

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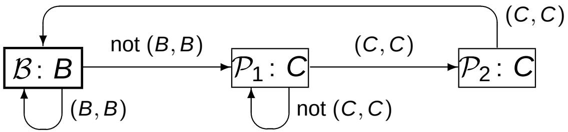

- Two-period punishment after deviation from (B,B)
- If both players choose C during punishment phase then after two periods they both revert to B
- If one player does not choose C in first period of punishment then punishment restarts

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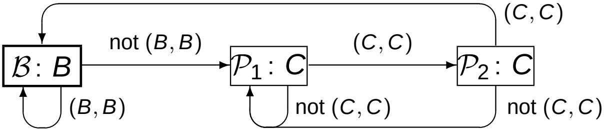

- Two-period punishment after deviation from (B, B)
- If both players choose C during punishment phase then after two periods they both revert to B
- If one player does not choose C in first period of punishment then punishment restarts
- Deviation from C in second period of punishment $\Rightarrow$ transition to first punishment state: punishment restarts

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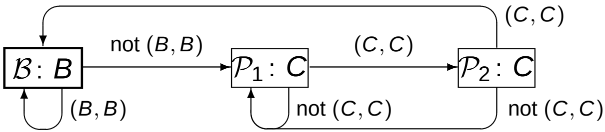

- Player is punished for not carrying out punishment

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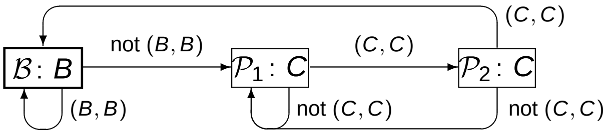

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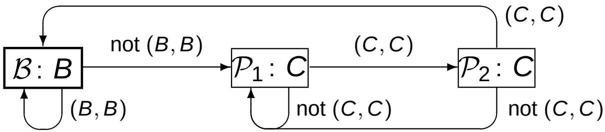

- Player is punished for not carrying out punishment
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- Suppose P2 adheres to strategy. Can P1 increase her payoff by deviating at the start of a subgame, holding rest of her strategy fixed?

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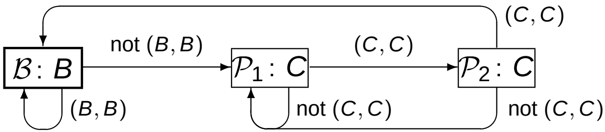

- Player is punished for not carrying out punishment
- SPE for both players to use this strategy?
- Suppose P2 adheres to strategy. Can P1 increase her payoff by deviating at the start of a subgame, holding rest of her strategy fixed?
- After any history, both players' automata are in same state, so need to consider only three cases

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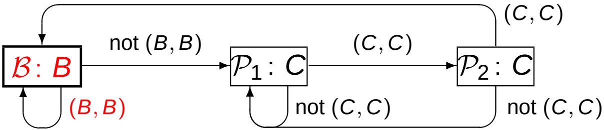

State $\mathcal{B}$

- P1 adheres to strategy $\Rightarrow$

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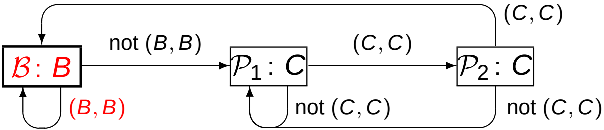

State $\mathcal{B}$

- P1 adheres to strategy $\Rightarrow$ payoffs $2, 2, 2, 2, 2, \ldots$

SPE of general infinitely repeated two-player games

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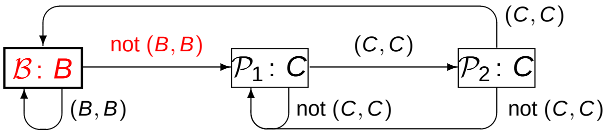

State $\mathcal{B}$

- P1 adheres to strategy $\Rightarrow$ payoffs $2,2,2,2,2,\ldots$
- P1 deviates $\Rightarrow$

SPE of general infinitely repeated two-player games

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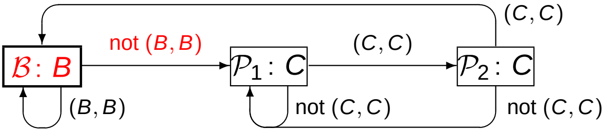

State $\mathcal{B}$

- P1 adheres to strategy $\Rightarrow$ payoffs $2, 2, 2, 2, 2, \ldots$
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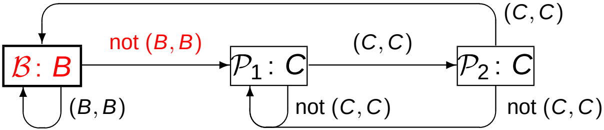

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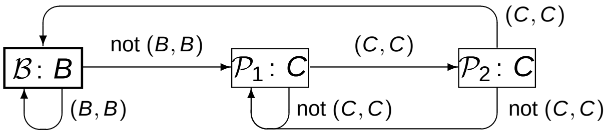

State $\mathcal{B}$

- P1 adheres to strategy $\Rightarrow$ payoffs $2, 2, 2, 2, 2, \ldots$
- P1 deviates $\Rightarrow$ payoffs $(3 \text{ or } 0), 0, 0, 2, 2, \ldots$
- So adhering to strategy is optimal if $2 + 2\delta + 2\delta^2 \geq 3$, or $\delta \geq \frac{1}{2}(\sqrt{3} - 1) \approx 0.366$

SPE of general infinitely repeated two-player games

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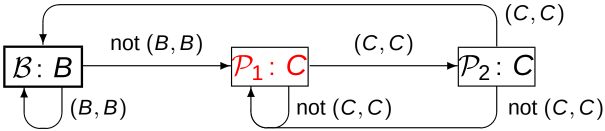

State $\mathcal{P}_1$

- P1 adheres to strategy $\Rightarrow$

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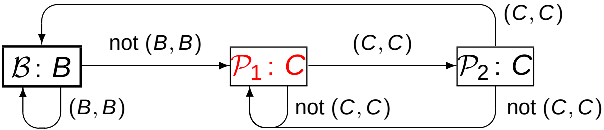

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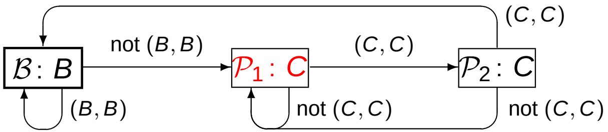

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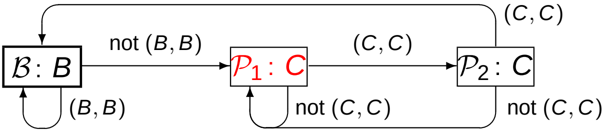

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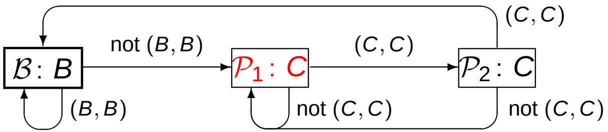

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SPE of general infinitely repeated two-player games

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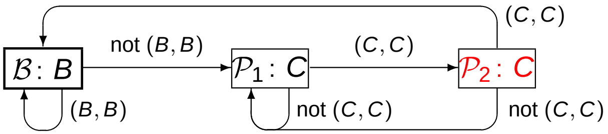

State $\mathcal{P}_2$

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	A	B	C
A	$4, 4$	$3, 0$	$1, 0$
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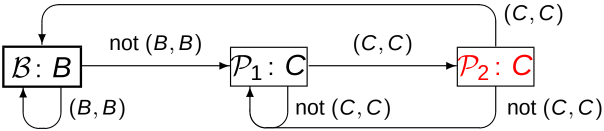

State $\mathcal{P}_2$

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SPE of general infinitely repeated two-player games

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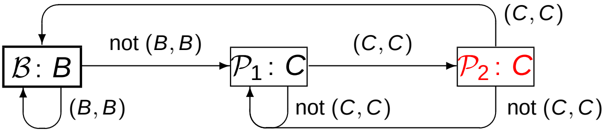

State $\mathcal{P}_2$

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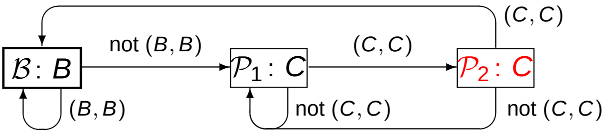

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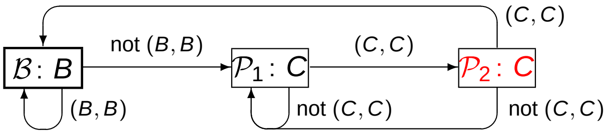

State $\mathcal{P}_2$

- P1 adheres to strategy $\Rightarrow$ payoffs $0, 2, 2, 2, 2, \ldots$
- P1 deviates $\Rightarrow$ payoffs $1, 0, 0, 2, 2, \ldots$
- So adhering to strategy is optimal if $2\delta + 2\delta^2 \geq 1$, or certainly if $2\delta^2 \geq 1$

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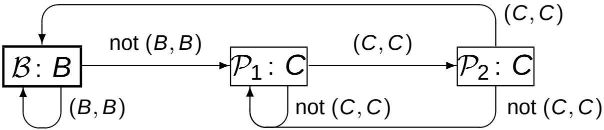

Conclusion

We have $\frac{1}{2}(\sqrt{3}-1) < \frac{1}{2}\sqrt{2}$, so strategy pair in which both players use this strategy is subgame perfect equilibrium if $\delta \geq \frac{1}{2}\sqrt{2} \approx 0.707$

SPE of general infinitely repeated two-player games

Idea behind example can be extended to any two-player game

Proposition (*Subgame perfect equilibrium folk theorem for two-player games*)

Every strictly enforceable payoff profile of a two-player strategic game G is (at least) arbitrarily close to a subgame perfect equilibrium payoff profile of the δ-discounted infinitely repeated game of G when δ is sufficiently close to 1.

SPE of general infinitely repeated two-player games

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Every strictly enforceable payoff profile of a two-player strategic game G is (at least) arbitrarily close to a subgame perfect equilibrium payoff profile of the δ-discounted infinitely repeated game of G when δ is sufficiently close to 1.

- Result can be extended to n-player games in which the set of feasible payoffs is n-dimensional (Proposition 151.1 in book)

Finitely repeated games: Nash equilibrium

Consider a game played a fixed finite number of times

Finitely repeated games: Nash equilibrium

Consider a game played a fixed finite number of times

Example: *Prisoner's Dilemma*

	C	D
C	$3,3$	$0,4$
D	$4,0$	$1,1$

Finitely repeated games: Nash equilibrium

Consider a game played a fixed finite number of times

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Claim: In every Nash equilibrium of finitely repeated *Prisoner's Dilemma* the outcome in every period is (D, D)

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Argument:

- Suppose outcome is not (D, D) in some period
- Let t be *last* period in which outcome is not (D, D) (because horizon is finite, such t exists)

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- Suppose outcome is not (D,D) in some period
- Let t be *last* period in which outcome is not (D,D) (because horizon is finite, such t exists)
- At least one player can profitably deviate from a^t—say P1
- Consider strategy of P1 that chooses profitable deviation in period t and D subsequently, regardless of history
- This strategy is profitable deviation in repeated game

Finitely repeated games: Nash equilibrium

- Result depends on special property of *Prisoner's Dilemma*: in unique Nash equilibrium, both players' payoffs are their minmax payoffs

Finitely repeated games: Nash equilibrium

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- For any strategic game G, outcome in last period of repeated game must be Nash equilibrium of G

Finitely repeated games: Nash equilibrium

- Result depends on special property of *Prisoner's Dilemma*: in unique Nash equilibrium, both players' payoffs are their minmax payoffs
- For any strategic game G, outcome in last period of repeated game must be Nash equilibrium of G
- *But* if G has Nash equilibrium in which some player's payoff exceeds her minmax payoff, earlier outcomes need not be Nash equilibria of G: deviant can be punished with minmax payoff

Finitely repeated games: Nash equilibrium

Example

	A	B	C
A	$3, 3$	$0, 4$	$1, 0$
B	$4, 0$	$2, 2$	$1, 0$
C	$0, 1$	$0, 1$	$0, 0$

Finitely repeated games: Nash equilibrium

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A	$3, 3$	$0, 4$	$1, 0$
B	$4, 0$	$2, 2$	$1, 0$
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- Unique NE: (B, B)

Finitely repeated games: Nash equilibrium

Example

	A	B	C
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- Unique NE: (B,B)
- Minmax payoffs: $(1,1)$

Finitely repeated games: Nash equilibrium

Example

	A	B	C
A	$3, 3$	$0, 4$	$1, 0$
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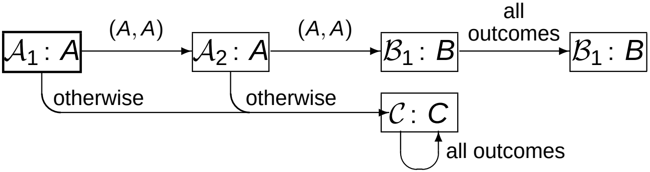
- Unique NE: (B, B)
- Minmax payoffs: $(1, 1)$

Consider 4-period game and suppose both players use strategy



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Example

	A	B	C
A	$3, 3$	$0, 4$	$1, 0$
B	$4, 0$	$2, 2$	$1, 0$
C	$0, 1$	$0, 1$	$0, 0$

- Unique NE: (B, B)
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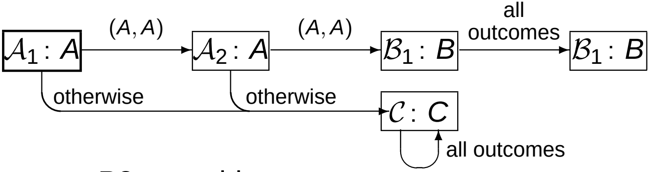
Consider 4-period game and suppose both players use strategy



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Finitely repeated games: Nash equilibrium

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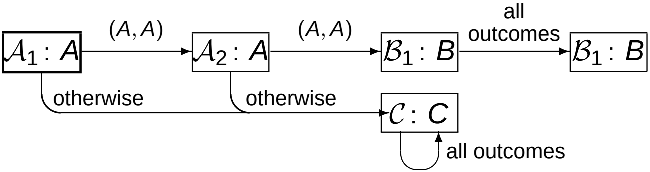

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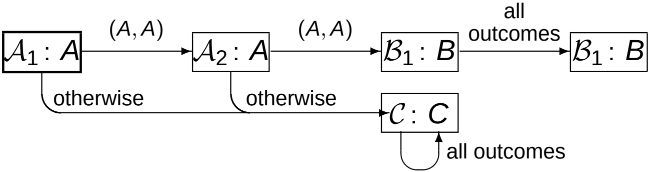

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- If P1 uses the strategy, outcome is $((A, A), (A, A), (B, B), (B, B))$, with payoffs $(10, 10)$
- If P1 deviates in period 1, payoff is at most $4 + 1 + 1 + 1 = 7$

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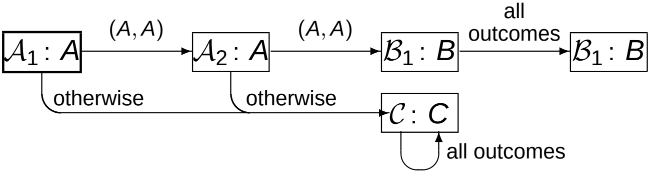

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- If P1 uses the strategy, outcome is $((A,A),(A,A),(B,B),(B,B))$, with payoffs $(10,10)$
- If P1 deviates in period 2, payoff is at most $3+4+1+1=9$

Finitely repeated games: Nash equilibrium

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A	$3,3$	$0,4$	$1,0$
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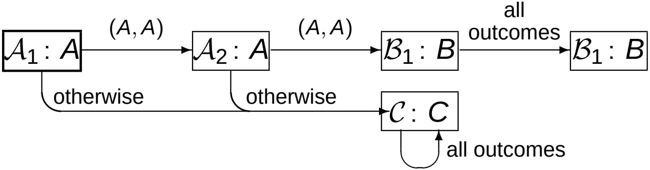

- Suppose P2 uses this strategy
- If P1 uses the strategy, outcome is $((A,A),(A,A),(B,B),(B,B))$, with payoffs $(10,10)$
- If P1 deviates in periods 3 or 4, she is worse off because (B,B) is NE of stage game

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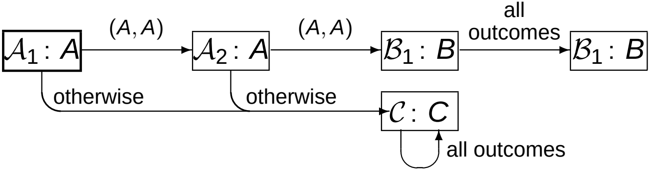

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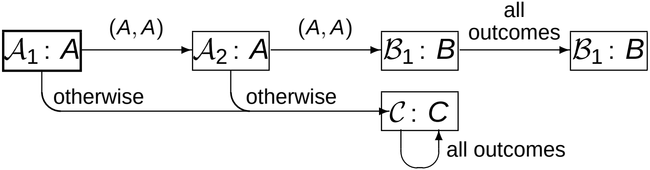

- Conclusion: strategy pair in which each player uses the strategy is a NE
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Finitely repeated games: Nash equilibrium

Proposition (*Nash folk theorem for finitely repeated games*)

If G has a Nash equilibrium in which the payoff of every player i exceeds her minmax payoff, then for any strictly enforceable outcome a^* of G and any $\varepsilon > 0$ there exists T^* such that if $T > T^*$ then the T-period repeated game of G has a Nash equilibrium in which the payoff of every player i is within ε of $u_i(a^*)$.

Finitely repeated games: Nash equilibrium

Proof

- For each player j, let p_{-j} be a list of actions of the other players that holds j's payoff to its minmax value, v_j:

$$p_{-j} \in \arg\min_{a_{-j} \in A_{-j}} \left(\max_{a_j \in A_j} u_j(a_{-j}, a_j) \right)$$

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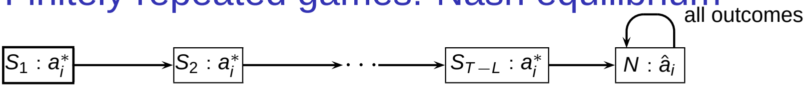
Finitely repeated games: Nash equilibrium



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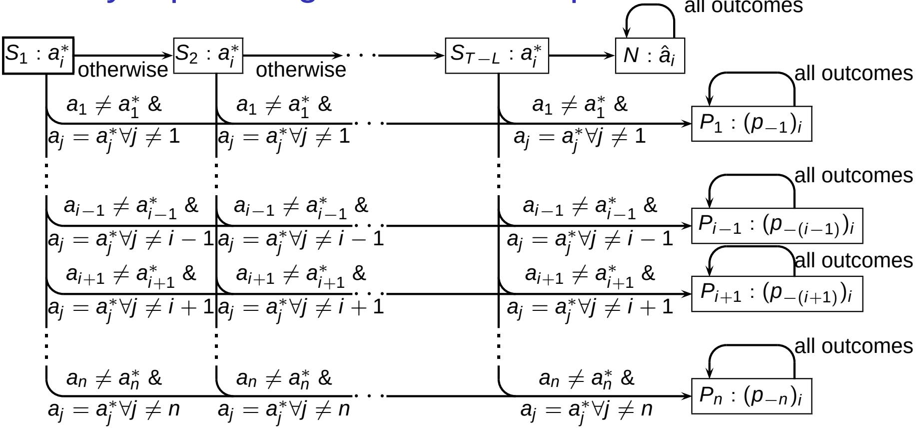

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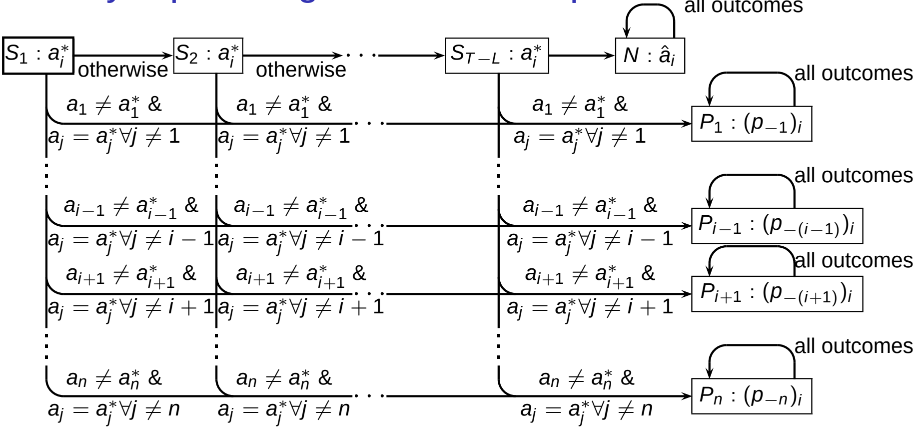

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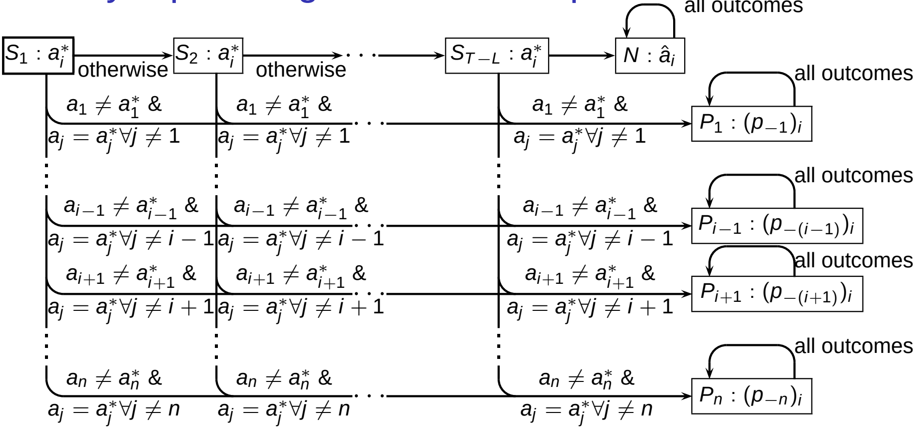

- Cannot profitably deviate by changing actions in last L periods because $\hat{a}$ is NE of G
- If L large enough, cannot profitably deviate by changing actions in earlier periods because $u_i(\hat{a})$ exceeds i's minmax payoff

Finitely repeated games: Subgame perfect equilibrium

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- If G has more than one Nash equilibrium payoff profile, punishment *is* possible

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If G has more than one Nash equilibrium payoff profile, credible punishment is possible

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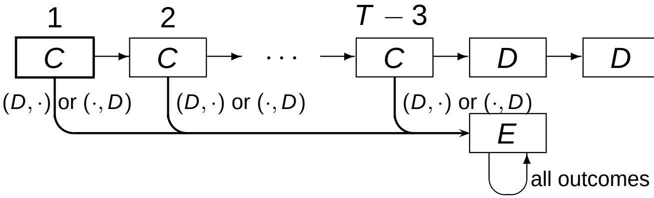

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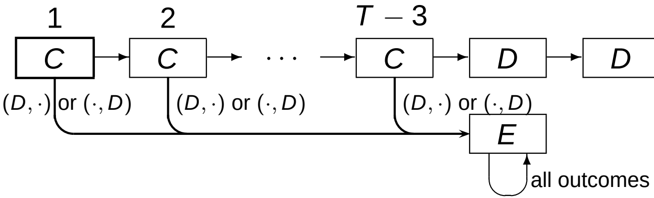

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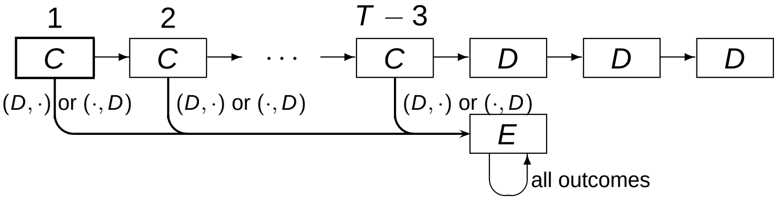

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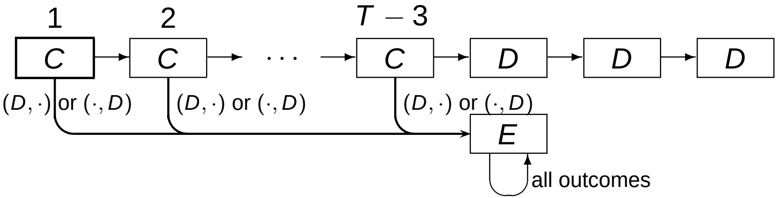

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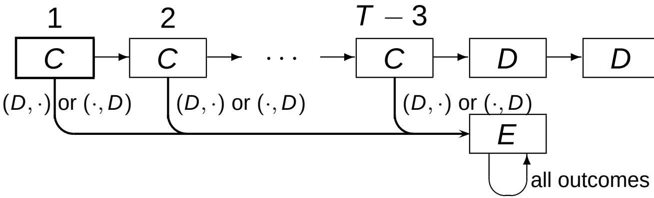

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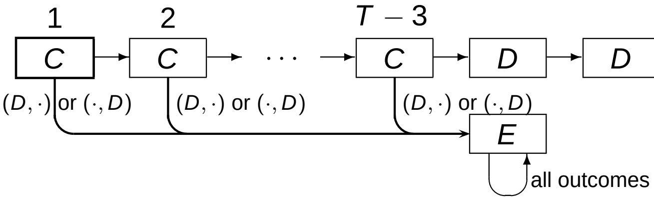

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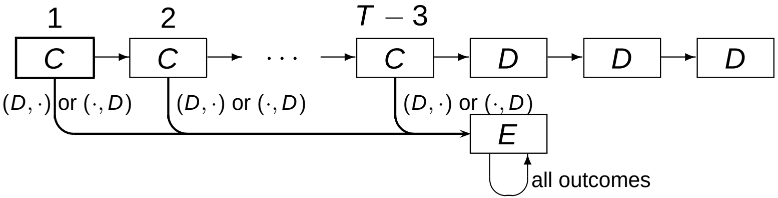

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- T large $\Rightarrow$ average payoffs approach 3

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- The strategies required to do so are more complicated than the ones in the example

Finitely repeated games: Subgame perfect equilibrium

Proposition (*Subgame perfect Folk theorem for finitely repeated games*)

Let a^* be a strictly enforceable outcome of the two-player game G. Assume that for each $i \in N$ there are two Nash equilibria of G that differ in the payoff of player i. Then for any $\varepsilon > 0$ there exists an integer T^* such that if $T > T^*$ the T-period repeated game of G has a subgame perfect equilibrium in which the payoff of each player i is within ε of $u_i(a^*)$.

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As in case of infinitely repeated games, extension to many players requires restriction on dimension of set of feasible payoff profiles

Repeated games: Summary

Infinitely repeated games with discounting

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If stage game has NE in which every player's payoff exceeds her minmax payoff, then for T large enough the set of average payoff profiles generated by Nash equilibria of T-period repeated game is essentially set of enforceable payoff profiles of stage game

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- Can model this dependence by allowing payoffs to depend on a state variable

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Total payoff for player i given states $(s^1, s^2, \ldots)$ and action profiles $(a^1, a^2, \ldots)$:

$$\sum_{t=1}^{\infty} \delta^{t-1} u_i(a^t, s^t)$$

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Can show that MPE in possibly mixed strategies exists if A and S are finite

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Consider infinitely repeated Prisoner's dilemma

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- $\Rightarrow$ unique MPE: each player always chooses D

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- Suggests that one should be cautious about whether MPE is an appropriate concept even outside of repeated games
- Typical justification of MPE is based on analytical convenience

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Two states, equally likely in each period *regardless of history*

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Does game have SPE with outcome (C, C) in every period?

- Consider pair of grim strategies: after any deviation, play D forever in both states
- After any history with no deviations, payoff from equilibrium strategy in state 2:

$$5 + \sum_{t=1}^{\infty} \delta^t \left(\tfrac{1}{2} \cdot 3 + \tfrac{1}{2} \cdot 5\right) = 5 + \frac{4\delta}{1-\delta}$$

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- At any history with at least one deviation, no incentive to deviate in either state

So game has SPE with outcome (C, C) in every period whenever $\delta \geq \frac{1}{4}$

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Failure to cooperate in one state makes it more difficult to sustain cooperation in the other one

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$\Rightarrow$ unique MPE involves both players choosing D in both states

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Two states, *identical* payoffs

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Begin in state 1; remain in state 1 as long as both choose C; switch to state 2 forever if either player chooses D in any period

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- Every subgame in state 2 is identical to repeated PD $\Rightarrow$ both players must choose D in MPE

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What are the MPE of this game?

- Markov strategy consists of a pair of actions: one to be chosen in state 1, the other in state 2
- Every subgame in state 2 is identical to repeated PD $\Rightarrow$ both players must choose D in MPE
- It one MPE, both choose D also in state 1

Example 3

Two states, *identical* payoffs

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- It one MPE, both choose D also in state 1
- But if both choose C in state 1, then strategies are equivalent to grim strategies $\Rightarrow$ MPE if $\delta \geq \frac{1}{3}$

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With one state, game has a unique MPE; with two identical states, game has two MPE, including one in which players do not play the NE of stage game

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- Some argue that states should be defined according to payoff-relevance: no two states should be identical in terms of payoffs
- But this assumption is not entirely satisfactory: with only a very small difference between payoffs in the two states, the example goes through (e.g., there could be an additional tiny benefit associated with the first defection)

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