Economics 2030

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Solutions to Problem Set 7

1. (a) The game is illustrated in Figure 1 in the case that n is odd.

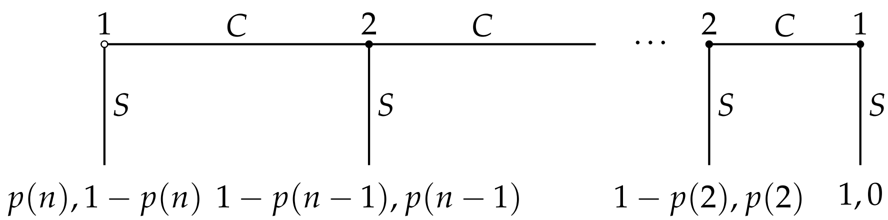

Figure 1. The game in Question 1.

(b) Use backward induction. The player whose turn it is to move when the players are 1 meter apart has no option but to shoot. The other player shoots when the players are 2 meters apart because $p(2) > 0$. When the players are 3 meters apart the player whose turn it is to move shoots if $p(3) > 1 - p(2)$. Let k^* be the largest integer for which $p(k^*) > 1 - p(k^* - 1)$. Then in the unique subgame perfect equilibrium the player whose turn it is to move when the players are k meters apart moves closer to the other player if $k > k^*$ and shoots otherwise.

(c) No, the game has no such Nash equilibrium. For a strategy pair in which player 1 shoots on her first move to be a Nash equilibrium, player 2 must shoot at her first move, otherwise player 1 can increase her payoff by taking a step on her first move and shooting on her second move. But if player 2 shoots on her first move, player 1 is better off taking a step on her first move because $1 - p(n-1) > p(n)$.

2. Consider a strategy profile in which each candidate chooses the median m of the citizens' favorite positions and the citizens' strategies are defined as follows.

- After a history in which every candidate chooses m, each citizen i votes for candidate j, where j is the smallest integer greater than or equal to in/q. (That is, the citizens split their votes equally among the n candidates. If there are 3 candidates and 15 citizens, for example, citizens 1 through 5 vote for candidate 1, citizens 6 through 10 vote for candidate 2, and citizens 11 through 15 vote for candidate 3.)

- After a history in which all candidates enter and every candidate but j chooses m, each citizen votes for candidate j if her favorite position is closer to j's position than it is to m, and for some candidate ℓ whose position is m otherwise. (All citizens who do not vote for j vote for the *same* candidate ℓ.)
- After any other history, the citizens' action profile is any Nash equilibrium of the voting subgame in which no citizen's action is weakly dominated.

Every such strategy profile generates an outcome in which all candidates enter and choose the median of the citizens' favorite positions, and tie for first place. After every history of one of the first two types, every citizen votes for one of the candidates who is closest to her favorite position, so no citizen's strategy is weakly dominated. After a history of the third type, no citizen's strategy is weakly dominated by construction.

Every such strategy profile is a subgame perfect equilibrium by the following argument.

In each voting subgame the citizens' strategy profile is a Nash equilibrium:

- after the history in which the candidates' positions are the same, equal to m, no citizen's vote affects the outcome
- after a history in which all candidates enter and every candidate but j chooses m, a change in any citizen's vote either has no effect on the outcome or makes it worse for her
- after any other history the citizens' strategy profile is a Nash equilibrium by construction.

Now consider the candidates' choices at the start of the game. If any candidate deviates by choosing a position different from that of the other candidates, she loses, rather than tying for first place. If any candidate deviates by staying out of the race, the outcome is worse for her than adhering to the equilibrium, and tying for first place. Thus each candidate's strategy is optimal given the other players' strategies.

[The claim that every voting subgame has a (pure) Nash equilibrium in which no citizen's action is weakly dominated, which you are not asked to prove, may be demonstrated as follows. Given the candidates' positions, choose the candidate, say j, ranked last by the smallest number of citizens. Suppose that all citizens except those who rank j last vote for j; distribute the votes of the citizens who rank j last as equally as possible among the other candidates. Each citizen's action is not weakly dominated (no citizen votes for the candidate she ranks last) and, given $q \geq 2n$, no change in any citizen's vote affects the outcome, so that the list of citizens' actions is a Nash equilibrium of the voting subgame.]

Source: Feddersen, Timothy J., Itai Sened, and Stephen G. Wright (1990), "Rational voting and candidate entry under plurality rule", *American Journal of Political Science* **34**, 1005–1016.

3. The following extensive game models the situation.

Players The seller and m buyers.

Histories $\varnothing$, the set of profiles $(p_1, \ldots, p_m)$, and the set of sequences of the form $((p_1, \ldots, p_m), j)$, where each p_i is a price (nonnegative number) and j is either 0 or one of the sellers (an integer from 1 to m), with the interpretation that p_i is the offer of buyer i, $j = 0$ means that the seller accepts no offer, and $j \geq 1$ means that the seller accepts buyer j's offer.

Player function $P(\varnothing)$ is the set of buyers and $P(p_1, \ldots, p_m)$ is the seller for every history $(p_1, \ldots, p_m)$.

Actions The set $A_i(\varnothing)$ of actions of buyer i at the start of the game is the set of prices (nonnegative numbers). The set $A_s(p_1, \ldots, p_m)$ of actions of the seller after the buyers have made offers is the set of integers from 0 to m.

Preferences Each player's preferences are represented by the payoffs given in the question.

To find the subgame perfect equilibria of the game, first consider the subgame following a history $(p_1, \ldots, p_m)$ of offers. The seller's best action is to accept the highest price, or one of the highest prices in the case of a tie.

I claim that a strategy profile is a subgame perfect equilibrium of the whole game if and only if the seller's strategy is the one just described, and among the buyers' strategies $(p_1, \ldots, p_m)$, every offer p_i is at most v and at least two offers are equal to v.

Such a strategy profile is a subgame perfect equilibrium by the following argument. If the buyer with whom the seller trades raises her offer then her payoff becomes negative, while if she lowers her offer she no longer trades and her payoff remains zero. If any other buyer raises her offer then either she still does not trade, or she trades at a price greater than v and hence receives a negative payoff.

No other profile of actions for the buyers at the start of the game is part of a subgame perfect equilibrium by the following argument.

- If some offer exceeds v then the buyer who submits the highest offer can induce a better outcome by reducing her offer to a value below v, so that either the seller does not trade with her, or, if the seller does trade with her, she trades at a lower price.
- If all offers are at most v and only one is equal to v, the buyer who offers v can increase her payoff by reducing her offer a little.
- If all offers are less than v then one of the buyers whose offer is not accepted can increase her offer to some value between the winning offer and v, induce the seller to trade with her, and obtain a positive payoff.

In any equilibrium the buyer who trades with the seller does so at the price v. Thus her payoff is zero. The other buyers do not trade, and hence also obtain the payoff of zero.