Economics 2030

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Solutions to Problem Set 8

1. The situation is modeled by the following extensive game.

   Players The parent and the child.

   Histories $\varnothing$, the set of actions a of the child, and the set of sequences (a, t), where a is an action of the child and t is a transfer from the parent to the child.

   Player function $P(\varnothing)$ is the child, $P(a)$ is the parent for every value of a.

   Preferences The child's preferences are represented by the payoff function $c(a) + t$ and the parent's preferences are represented by the payoff function $\min\{p(a) - t, c(a) + t\}$.

   To find the subgame perfect equilibria of this game, first consider the parent's optimal actions in the subgames of length 1. Consider the subgame following the choice of a by the child. We have $p(a) > c(a)$ (by assumption), so if the parent makes no transfer her payoff is $c(a)$. If she transfers \$1 to the child then her payoff increases to $c(a) + 1$. As she increases the transfer her payoff increases until $p(a) - t = c(a) + t$; that is, until $t = \frac{1}{2}(p(a) - c(a))$. (If she increases the transfer any more, she has less money than her child.) Thus the parent's optimal action in the subgame following the choice of a by the child is $t = \frac{1}{2}(p(a) - c(a))$.

   Now consider the whole game. Given the parent's optimal action in each subgame, a child who chooses a receives the payoff $c(a) + \frac{1}{2}(p(a) - c(a)) = \frac{1}{2}(p(a) + c(a))$. Thus in a subgame perfect equilibrium the child chooses the action that maximizes $p(a) + c(a)$, the sum of her own private income and her parent's income.

   Source: Becker, Gary S. (1974), "A theory of social interactions", *Journal of Political Economy* **82**, 1063–1093.

2. (a) A subgame perfect equilibrium:

- If player 1 demands less than 50, player 2 accepts her offer and player 1 is worse off.
- If player 1 demands more than 50, player 2 rejects her offer and proposes $(50, 50)$, which player 1 accepts, yielding her a payoff of 49.
- If player 1 rejects an offer of 49 or more she gets at most 50 with one period of delay, for a net payoff of 49.

The argument for player 2 is exactly the same.

(b) A subgame perfect equilibrium:

- If player 1 demands less, player 2 accepts her offer and player 1 is worse off.
- If player 1 rejects an offer of 99 or more she gets at most 100 with one period of delay, for a net payoff of 99.
- If player 2 demands less than 1, player 1 accepts her offer and she is worse off.
- If player 2 demands more than 1, player 1 rejects her offer and proposes $(100, 0)$, which player 2 accepts, yielding her a payoff -1.
- If player 2 rejects any offer, she gets 1 in the next period, yielding her a payoff of 0.

(c) No. Any strategy pair that yields such an outcome entails player 1's proposing $(0, 100)$ at the start of the game. For this proposal to be optimal, player 2 must reject any offer that gives her less than 100. But her rejecting an offer of more than 99 is not optimal because the best outcome of such a rejection is that she gets 100 with one period of delay, yielding her a payoff of 99.

3. Any subgame following a history that ends in player 3's approaching player 2 and player 2's deciding to stay with player 1 is a standard bargaining game of alternating offers with player 2 the first mover, and hence has the standard subgame perfect equilibrium, in which player 2's payoff is $1/(1+\delta)$ and player 1's payoff is $\delta/(1+\delta)$.

   Any subgame following a history that ends in player 3's approaching player 2 and player 2's deciding to bargain with player 3 is a standard bargaining game of alternating offers with player 3 the first mover and a pie of size k. Thus any such subgame has the standard subgame perfect equilibrium, in which player 3's payoff is $k/(1+\delta)$ and player 2's payoff is $\delta k/(1+\delta)$.

Now consider a subgame following a history that ends with player 3's approaching player 2. At the start of such a subgame, player 2 has a choice of either continuing with player 1, in which case she obtains the payoff $1/(1+\delta)$ (discounted to the first period of the subgame), or of switching to player 3, in which case she obtains the payoff $\delta k/(1+\delta)$ (discounted to the first period of the subgame). There are two main cases. (I ignore the case in which $k = 1/\delta$.)

$k < 1/\delta$ In this case, player 2 is better off staying with player 1 if approached by player 3. Thus in a subgame perfect equilibrium player 2 always stays with player 1, player 3 either approaches player 2 or does not, and players 1 and 2 make the same offers and use the same acceptance rules as they do in the standard bargaining game of alternating offers (with a pie of size 1).

$k > 1/\delta$ In this case, player 2 is better off switching to player 3 if approached by her. Player 1's payoff is 0 if player 2 deserts her, so she wants to avoid this outcome. To ensure that player 2 continues bargaining with her, she needs to offer her at least $\delta^2 k/(1+\delta)$ after any history in which player 3 has no approached player 2 (so that player 3 is still in the game). Given $k > 1/\delta$, this amount exceeds the amount she offers in a subgame perfect equilibrium of game in which player 3 is absent. Thus the game has a subgame perfect equilibrium in which after any history in which player 3 has never approached player 2,

- player 1 proposes $(1 - \delta^2 k/(1+\delta), \delta^2 k/(1+\delta))$ and accepts an offer y if and only if $y_1 \geq \delta(1 - k\delta^2/(1+\delta))$
- player 2 proposes $(\delta(1 - k\delta^2/(1+\delta), 1 - \delta(1 - k\delta^2/(1+\delta))$ and accepts an offer x if and only if $x_2 \geq \delta^2 k/(1+\delta)$ and rejects all approaches from player 3
- player 3's approaches player 2 whenever she has the opportunity to do so
- player 2 chooses to bargain with player 3 when given the opportunity.

4. Consider the subgame following the choice of e by player 1. This subgame is the bargaining game of alternating offers, so that it has a unique subgame perfect equilibrium, in which player 1 proposes $(e/(1+\delta), \delta e/(1+\delta))$ in the first period, and this offer is accepted by player 2. Player 1's payoff in the subgame is $e/(1+\delta) - e^2$. In the

first period she chooses e to maximize this payoff. Thus she chooses $e = 1/(2(1+\delta))$.

The sum of the equilibrium payoffs is

$$\frac{1}{2(1+\delta)} - \left(\frac{1}{2(1+\delta)}\right)^2 = \frac{1+2\delta}{(2(1+\delta))^2} < \tfrac{1}{4}.$$

If player 1 chooses e, the sum of the payoffs is $e - e^2$, whose maximal value is $\frac{1}{4}$, attained when $e = \frac{1}{2}$.

5. I claim that there are proposals x^*, y^*, and z^* such that the game has a subgame perfect equilibrium in which

   - player 1 always proposes x^*, accepts an offer z from player 3 if and only if $z_1 \geq z_1^*$, and accepts an offer y from player 2 if and only if $y_1 \geq y_1^*$
   - player 2 always proposes y^*, accepts an offer x from player 1 if and only if $x_2 \geq x_2^*$, and accepts an offer z from player 3 if and only if $z_2 \geq z_2^*$
   - player 3 always proposes z^*, accepts an offer y from player 2 if and only if $y_3 \geq y_3^*$, and accepts an offer x from player 1 if and only if $x_3 \geq x_3^*$.

   We may reasonably guess that in a subgame perfect equilibrium each responder is indifferent between accepting and rejecting the offer she faces in. This indifferent means that the proposals must satisfy the following conditions.

$$\begin{aligned}
z_1^* &= \delta x_1^* \\
z_2^* &= \delta x_2^* \\
x_2^* &= \delta y_2^* \\
x_3^* &= \delta y_3^* \\
y_3^* &= \delta z_3^* \\
y_1^* &= \delta z_1^*.
\end{aligned}$$

These equations have the following unique solution.

$$
\begin{aligned}
(x_1^*, x_2^*, x_3^*) &= \left(\frac{1}{1+\delta+\delta^2}, \frac{\delta}{1+\delta+\delta^2}, \frac{\delta^2}{1+\delta+\delta^2}\right) \\
(y_1^*, y_2^*, y_3^*) &= \left(\frac{\delta^2}{1+\delta+\delta^2}, \frac{1}{1+\delta+\delta^2}, \frac{\delta}{1+\delta+\delta^2}\right) \\
(z_1^*, z_2^*, z_3^*) &= \left(\frac{\delta}{1+\delta+\delta^2}, \frac{\delta^2}{1+\delta+\delta^2}, \frac{1}{1+\delta+\delta^2}\right).
\end{aligned}
$$

To show that the strategy profile defined by these equations is a subgame perfect equilibrium, we may use the one deviation property. The argument for each player is similar. I present the details for player 1.

Consider a history after which player 1 makes an offer. If she offers both player 2 and player 3 at least as much as her strategy profile dictates, and at least one of them more, then they both accept the offer and player 1 is worse off. If she reduces the amount she gives to at least one of these players, her offer is rejected and she obtains y_1^* with one period of delay, which is worth less than x_1^* immediately.

Now consider a history after which player 1 responds to an offer of player 3. If she rejects z^* then she makes the counteroffer x^* in the next period, and this offer is accepted. Thus her payoff is $\delta_1 x_1^*$, which is equal to her payoff z_1^* when she accepts z^*.

Finally consider a history after which player 1 responds to an offer of player 2. If she rejects y^* then player 3 makes the counteroffer z^* in the next period, and this offer is accepted. Thus her payoff is $\delta_1 z_1^*$, which is equal to her payoff y_1^* when she accepts y^*.

We conclude that the strategy profile is a subgame perfect equilibrium.