ECO2030: Microeconomic Theory II, module 1

Lecture 10

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2018.11.29

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Repeated games

- *Same* set of players interact repeatedly
- Every player remembers other players' previous actions
- Each player can condition her action in period t on other players' actions in periods $1, \ldots, t-1$
- Extensive game with perfect information and simultaneous moves

Infinitely repeated games

Let $G = \langle N, (A_i), (\succsim_i) \rangle$ be strategic game; denote $A = \times_{i \in N} A_i$

An *infinitely repeated game* of G is an extensive game $\langle N, H, P, (\succsim_i^*) \rangle$ where

- $H = \{\varnothing\} \cup (\cup_{t=1}^{\infty} A^t) \cup A^\infty$ (where A^∞ is set of infinite sequences $(a^t)_{t=1}^{\infty}$ of action profiles in G)
- $P(h) = N$ for all h
- $\succsim_i^*$ is a preference relation on A^∞ that extends $\succsim_i$ in the sense that if $(a^t) \in A^\infty$, $a \in A$, $a' \in A$, and $a \succsim_i a'$ then

$$(a^1, \ldots, a^{t-1}, a, a^{t+1}, \ldots) \succsim_i^* (a^1, \ldots, a^{t-1}, a', a^{t+1}, \ldots)$$

for all values of t

Repeated games: Example

Suppose G is *Prisoner's Dilemma*

	C	D
C	$3,3$	$0,4$
D	$4,0$	$1,1$

G has unique Nash equilibrium, (D,D)

Repeated game

"[T]he strategies: [player 1] plays [C] 'til [player 2] plays [D], then [D] ever after, [player 2] plays [C] 'til [player 1] plays [D], then [D] ever after, are very nearly at equilibrium [in a 100-period repetition of the game] and in a game with an indeterminate stop point or an infinite game with interest on utility it is an equilibrium point." (John F. Nash, commenting on an experiment in January 1950)

Repeated games: Example

Suppose G is *Prisoner's Dilemma*

	C	D
C	$3,3$	$0,4$
D	$4,0$	$1,1$

G has unique Nash equilibrium, (D,D)

Infinitely repeated game

- Define strategy s_i^* by $s_i^*(\varnothing) = C$ and

$$s_i^*(a^1,\ldots,a^{t-1}) = \begin{cases} C & \text{if } a_j^\tau = C \text{ for } \tau = 1,\ldots,t-1 \\ D & \text{otherwise} \end{cases}$$

where j is the other player

Repeated games: Example

	C	D
C	3, 3	0, 4
D	4, 0	1, 1

$$s_i^*(a^1, \ldots, a^{t-1}) = \begin{cases} C & \text{if } a_j^\tau = C \text{ for } \tau = 1, \ldots, t-1 \\ D & \text{otherwise} \end{cases}$$

If P2 uses strategy s_2^*, what is P1's best response?

- Strategy that chooses C after every history in which P2 chose C in every period (e.g. s_1^*)
  - outcome (C, C) in every period
  - payoffs $(3, 3)$ in every period
- Strategy that chooses D in some period t after history in which P2 chose C in every previous period
  - outcome in period t is (D, C), with payoffs $(4, 0)$
  - in every subsequent period P2 chooses D
  - payoff to P1 in every subsequent period is at most 1
  - best option for P1 is to choose D after period t

Repeated games: Example

	C	D
C	3, 3	0, 4
D	4, 0	1, 1

$$s_i^*(a^1, \ldots, a^{t-1}) = \begin{cases} C & \text{if } a_j^\tau = C \text{ for } \tau = 1, \ldots, t-1 \\ D & \text{otherwise} \end{cases}$$

- So P1's choice is between

$$\text{stick to } C \Rightarrow \text{payoffs } (3, 3, \ldots, 3, 3, 3, 3, \ldots)$$
$$\text{deviate to } D \text{ in period } t \Rightarrow \text{payoffs } (3, 3, \ldots, 3, 4, 1, 1, \ldots)$$

- If P1 is not too impatient, $(3, 3, \ldots)$ is better, so best response is strategy that chooses C after every history in which P2 chooses C in every period
- s_1^* is such a strategy
- Argument is symmetric for P2, so if players are sufficiently patient, (s_1^*, s_2^*) is a Nash equilibrium

Repeated games: Example

	C	D
C	$3, 3$	$0, 4$
D	$4, 0$	$1, 1$

Conclusion

If players sufficiently patient, strategy pair (s_1^*, s_2^*) is Nash equilibrium of infinitely repeated game, where $s_i^*(\varnothing) = C$ and

$$s_i^*(a^1, \ldots, a^{t-1}) = \begin{cases} C & \text{if } a_j^\tau = C \text{ for } \tau = 1, \ldots, t-1 \\ D & \text{otherwise} \end{cases}$$

where j is the other player

Outcome of this equilibrium is (C, C) in every period

Repeated games: Questions

- What do we mean by "patience"?
- How patient do the players have to be for the strategy pair (s_1^*, s_2^*) to be a Nash equilibrium?
- Can the outcome path in which (C, C) is played in every period be supported with less severe punishments?
- What outcomes other than (C, C) in every period are supported?
- What about subgame perfect equilibria rather than Nash equilibria? Is it optimal for each player to punish the other player for deviating?
- What happens in games other than *Prisoner's Dilemma*?

Preferences in repeated games

Discounting Represented by discounted sum of one-shot payoffs: sequence $(a^1, a^2, \ldots)$ of outcomes has payoff

$$\sum_{t=1}^{\infty} \delta^{t-1} u_i(a^t)$$

Limit of means Preferences essentially represented by

$$\lim_{T \to \infty} \frac{\sum_{t=1}^{T} u_i(a^t)}{T}$$

though need to deal with possibility that limit doesn't exist

Overtaking Won't discuss

Preferences in repeated games

- Will concentrate on preferences with discounting
- Two strategic games generate same preferences with discounting in repeated game $\Leftrightarrow$ each player's payoffs in one game are affine transformation of her payoffs in other game

Example

	C	D
C	$3, 3$	$0, 4$
D	$4, 0$	$1, 1$

G: payoffs (u_1, u_2)

	C	D
C	$7, 8$	$1, 11$
D	$9, -1$	$3, 2$

G': payoffs (v_1, v_2)

- Payoffs of P1: $v_1(a) = 1 + 2u_1(a)$ for all a
- Payoffs of P2: $v_2(a) = -1 + 3u_2(a)$ for all a
- So *preferences* of player i in infinitely repeated game of G are same as preferences of player i in G', for $i = 1, 2$

Preferences in repeated games

- Will concentrate on preferences with discounting
- Two strategic games generate same preferences with discounting in repeated game $\Leftrightarrow$ each player's payoffs in one game are affine transformation of her payoffs in other game

Example

	C	D
C	$3,3$	$0,4$
D	$4,0$	$1,1$

G: payoffs (u_1, u_2)

	C	D
C	$3,3$	$0,7$
D	$7,0$	$1,1$

G': payoffs (u_1, u_2)

- Payoffs of P1 are ordinally same in two games but *not* affine transforms of each other
- Different preferences in repeated game: for δ close to one

$$((C,C),(C,C)) \succ_1 ((C,D),(D,C)) \text{ for left game}$$

$$((C,C),(C,C)) \prec_1 ((C,D),(D,C)) \text{ for right game}$$

Preferences with discounting

- Instead of working with discounted sum, sometimes convenient to work with discounted average

$$(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}u_i(a^t)$$

- For constant stream of payoffs $(c, c, \ldots)$, discounted average is

$$(1-\delta)(c+\delta c+\delta^2 c+\ldots)=(1-\delta)\frac{c}{1-\delta}=c$$

- Sometimes refer to player's discounted average payoff simply as her payoff in repeated game

Nash equilibrium of *Prisoner's Dilemma*

Can now answer question: For the *Prisoner's Dilemma*

	C	D
C	$3, 3$	$0, 4$
D	$4, 0$	$1, 1$

how patient do the players have to be for the strategy pair (s_1^*, s_2^*) to be a Nash equilibrium, where

$$s_i^*(\varnothing) = C$$

$$s_i^*(a^1, \dots, a^{t-1}) = \begin{cases} C & \text{if } a_j^\tau = C \text{ for } \tau = 1, \dots, t-1 \\ D & \text{otherwise} \end{cases}$$

for $i = 1$, 2 and j is the other player?

Nash equilibrium of *Prisoner's Dilemma*

	C	D
C	$3, 3$	$0, 4$
D	$4, 0$	$1, 1$

Suppose P2 uses s_2^*

- P1 uses s_1^*
  - $\Rightarrow$ P1's payoff is $3 + 3\delta + 3\delta^2 + \cdots$
- P1 deviates
  - $\Rightarrow$ Either outcome remains same *or* changes to (D, C) in some period t
  - $\Rightarrow$ If outcome changes, P2 chooses D in every period $\geq t + 1$
  - $\Rightarrow$ Best strategy of P1 that deviates in t chooses D in every period $\geq t + 1$
  - $\Rightarrow$ P1's payoff is $3 + 3\delta + \cdots + 3\delta^{t-2} + 4\delta^{t-1} + \delta^t + \delta^{t+1} + \cdots$

Nash equilibrium of *Prisoner's Dilemma*

- P1 has no profitable deviation if and only if

$$3+3\delta+\cdots+3\delta^{t-2}+4\delta^{t-1}+\delta^{t}+\delta^{t+1}+\cdots\leq 3+3\delta+3\delta^{2}+\cdots$$

or

$$4\delta^{t-1}+\delta^{t}+\delta^{t+1}+\cdots\leq 3\delta^{t-1}+3\delta^{t}+3\delta^{t+1}+\cdots$$

or

$$4+\delta+\delta^{2}+\cdots\leq 3+3\delta+3\delta^{2}+\cdots$$

or

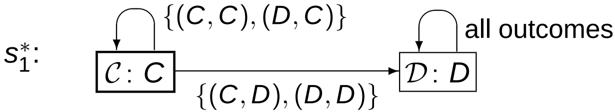
$$4+\frac{\delta}{1-\delta}\leq\frac{3}{1-\delta}\quad\Leftrightarrow\quad 4(1-\delta)+\delta\leq 3\quad\Leftrightarrow\quad\delta\geq\frac{1}{3}$$

- s_1^* is a best response of P1 to $s_2^* \Leftrightarrow \delta \geq \frac{1}{3}$

Conclusion: (s_1^*, s_2^*) is a Nash equilibrium if and only if $\delta \geq \frac{1}{3}$

Describing strategies

Can represent strategies compactly in figure:



Viewed this way, the strategy is an automaton, consisting of

a set Q_i (states)	$\{\mathcal{C}, \mathcal{D}\}$
$q_i^0 \in Q_i$ (initial state)	$\mathcal{C}$
$f_i : Q_i \to A_i$ (output function)	$f_i(\mathcal{C}) = C$, $f_i(\mathcal{D}) = D$
$\tau_i : Q_i \times A \to Q_i$ (transition function)	$\tau_i(\mathcal{C},(C,C)) = \tau_i(\mathcal{C},(D,C)) = \mathcal{C}$, $\tau_i(\mathcal{C},(C,D)) = \tau_i(\mathcal{C},(D,D)) = \tau_i(\mathcal{D},(a_1,a_2)) = \mathcal{D}$ for all (a_1,a_2)

Additional benefit of representing strategy in this way: measure of complexity of strategy is number of states

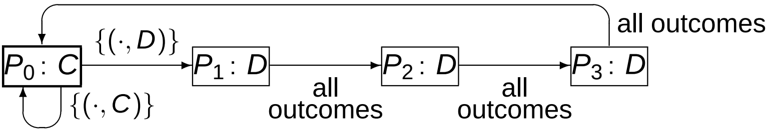
Describing strategies

Any automaton $\langle Q_i, q_i^0, f_i, \tau_i \rangle$ defines a strategy as follows:

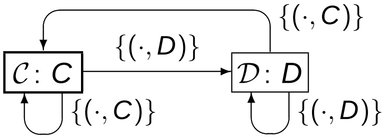
- $s_i(\emptyset) = f_i(q_i^0)$
- $s_i(a^1) = f_i(\tau_i(q_i^0, a^1))$ for all $a^1 \in A$
- $s_i(a^1, a^2) = f_i(\tau_i(\tau_i(q_i^0, a^1), a^2))$ for all $(a^1, a^2) \in A \times A$
- and so on

Strategies: Examples

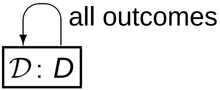
Three-period punishment



Tit-for-tat

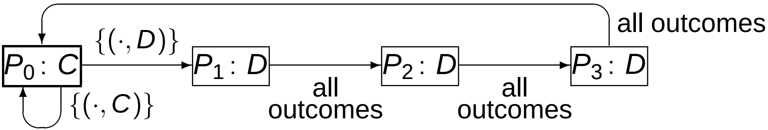


All-D



Nash equilibrium with limited punishment?

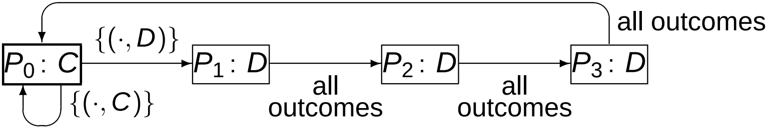
	C	D
C	$3, 3$	$0, 4$
D	$4, 0$	$1, 1$



- Does infinitely repeated game have Nash equilibrium in which each player uses limited punishment strategy?
- Suppose P2 uses limited punishment strategy with k periods of punishment
- When is it optimal for P1 to use same strategy?

Nash equilibrium with limited punishment?

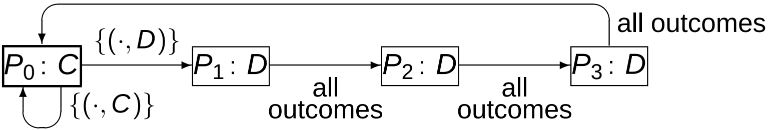
	C	D
C	$3, 3$	$0, 4$
D	$4, 0$	$1, 1$



- If P1 uses same strategy, outcome is (C, C) in every period $\Rightarrow$ P1's payoff is 3 in every period

Nash equilibrium with limited punishment?

	C	D
C	$3,3$	$0,4$
D	$4,0$	$1,1$

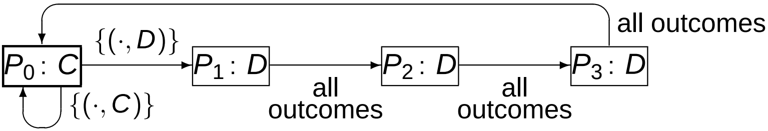


- If P1 has profitable deviation, then deviation to D in period 1 that returns to C in period $k+2$ is profitable
  Resulting outcomes and payoffs to P1:

(D, C)	4	
(D, D)	1	first period of punishment
(D, D)	1	
$\vdots$	$\vdots$	
(D, D)	1	last period of punishment
(C, C)	3	
(C, C)	3	

Nash equilibrium with limited punishment?

	C	D
C	$3,3$	$0,4$
D	$4,0$	$1,1$



- For deviation not to be profitable, need

$$4 + \delta + \delta^2 + \cdots + \delta^k \leq 3 + 3\delta + 3\delta^2 + \cdots + 3\delta^k$$

$$3 + (1 + \delta + \delta^2 + \cdots + \delta^k) \leq 3(1 + \delta + \delta^2 + \cdots + \delta^k)$$

$$3 \leq \frac{2(1 - \delta^{k+1})}{1 - \delta}$$

$$1 - 3\delta + 2\delta^{k+1} \leq 0$$

$k = 1 \Rightarrow \delta \geq \frac{1}{2}$; $k \uparrow \Rightarrow$ cutoff value of $\delta \downarrow \frac{1}{3}$

Nash equilibrium with limited punishment?

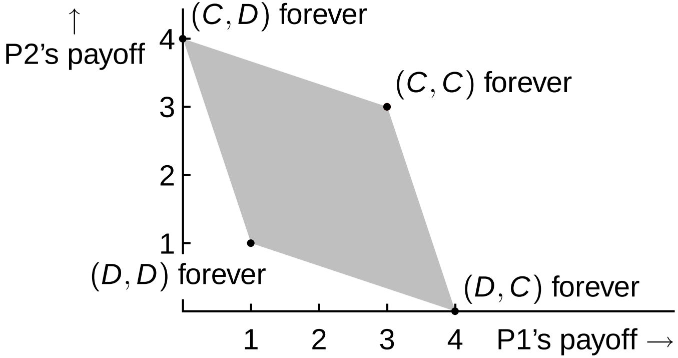
Conclusion

- For any value of $k \geq 1$, strategy pair in which each player punishes other for k periods in event of deviation is Nash equilibrium of infinitely repeated game if δ is close enough to 1
- Larger $k \Rightarrow$ smaller lower bound on δ: mutually desirable outcome (C, C) is sustained by short punishment only if players are relatively patient

What payoffs can be achieved in a Nash equilibrium?

Feasible payoffs

First consider payoffs that are *possible* when players are patient (discounting with δ close to 1, or limit of means)

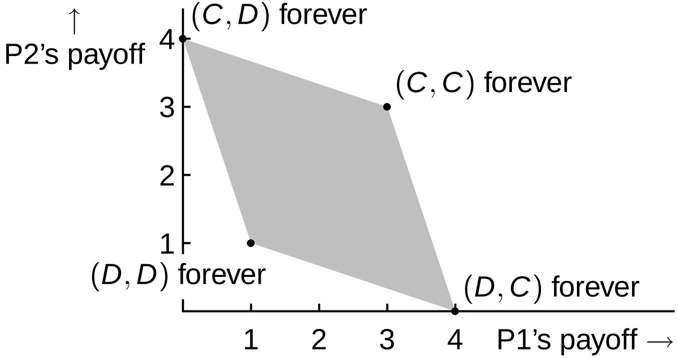


- Action pair (C, C) in every period $\Rightarrow$ (discounted average, and limit of means) payoffs $(3, 3)$

What payoffs can be achieved in a Nash equilibrium?

Feasible payoffs

First consider payoffs that are *possible* when players are patient (discounting with δ close to 1, or limit of means)

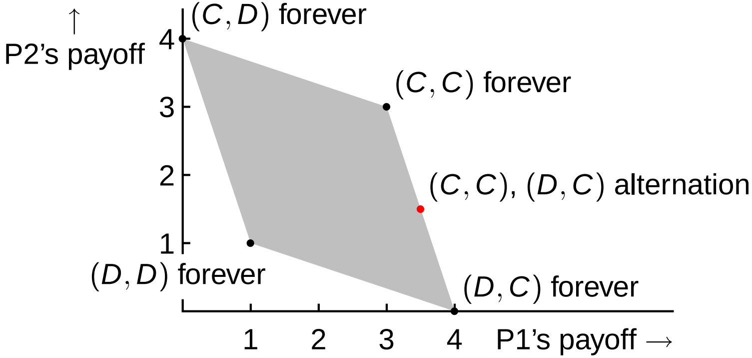


- Similarly for repetitions of other action pairs

What payoffs can be achieved in a Nash equilibrium?

Feasible payoffs

First consider payoffs that are *possible* when players are patient (discounting with δ close to 1, or limit of means)

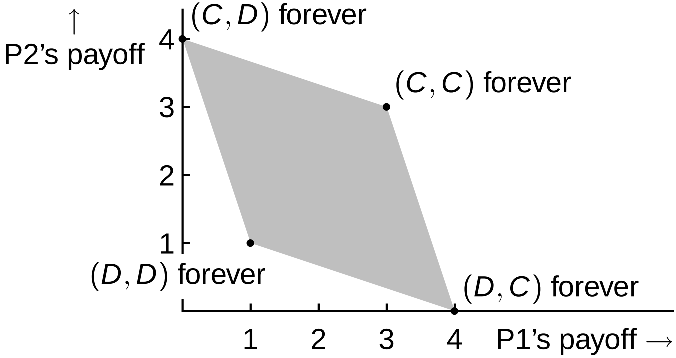


- Could alternate between (C, C) and (D, C) $\Rightarrow$ payoffs close to $(\frac{7}{2}, \frac{3}{2})$

What payoffs can be achieved in a Nash equilibrium?

Feasible payoffs

First consider payoffs that are *possible* when players are patient (discounting with δ close to 1, or limit of means)

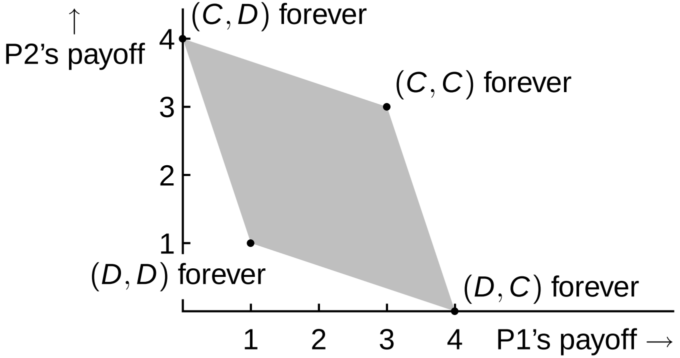


- Similarly could cycle through any other sequence of outcomes $\Rightarrow$ average of payoffs to outcomes in sequence

What payoffs can be achieved in a Nash equilibrium?

Feasible payoffs

First consider payoffs that are *possible* when players are patient (discounting with δ close to 1, or limit of means)



- Can approximately achieve any linear combination of payoffs in stage game

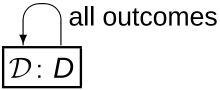
Nash equilibrium payoffs

One Nash equilibrium

Pair (s^*, s^*) of punishment strategies is a Nash equilibrium of repeated game, yielding discounted average payoffs $(3, 3)$

Another Nash equilibrium

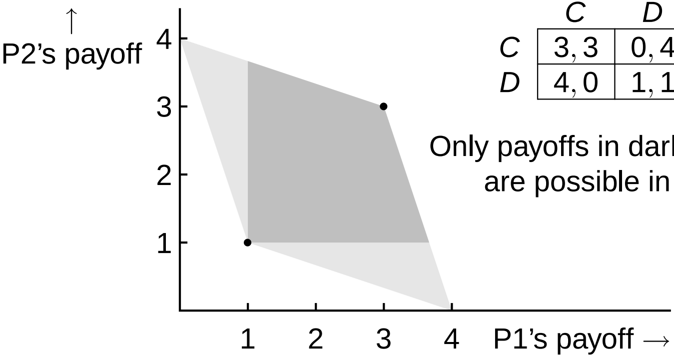
Consider strategy $\hat{s}$ in which each player chooses D after every history:



$(\hat{s}, \hat{s})$ is a Nash equilibrium of the repeated game, yielding payoffs $(1, 1)$

Nash equilibrium payoffs

These equilibria yield payoffs of $(3, 3)$ and $(1, 1)$. What other payoffs are possible in Nash equilibria?



	C	D
C	$3, 3$	$0, 4$
D	$4, 0$	$1, 1$

Only payoffs in dark region are possible in NE

- Are payoffs $(0, 4)$ possible in Nash equilibrium?
- No, because by choosing D after every history, P1 *guarantees* payoff of at least 1 in every period
- In any Nash equilibrium, payoff of each player is at least 1

Nash equilibrium payoffs

For general strategic game, the payoff player i can guarantee in any period is her minmax payoff

$$v_i = \min_{a_{-i} \in A_{-i}} \left(\max_{a_i \in A_i} u_i(a_{-i}, a_i) \right)$$

Example

	A	B	C
A	$1, 1$	$0, 0$	$2, 3$
B	$0, 0$	$1, 2$	$1, 2$
C	$0, 2$	$2, 3$	$3, 1$

- $v_1 = 1$
- $v_2 = 2$
- Note that pair of minmax actions (B, A) is not a Nash equilibrium of this game
- For *Prisoner's Dilemma*, $v_1 = v_2 = 1$, and pair of minmax actions, (C, C), *is* a Nash equilibrium

Nash equilibrium payoffs

- $w_i \geq v_i$ for all $i \in N \Rightarrow w$ is enforceable
- $w_i > v_i$ for all $i \in N \Rightarrow w$ is strictly enforceable

Proposition

For any strategic game G and any discount factor δ, every Nash equilibrium payoff profile of the δ-discounted infinitely repeated game of G is an enforceable payoff profile of G

Idea of proof

Every player i can get at least v_i in every period by choosing an action in the period that best responds to the other players' actions

Proof that every Nash equilibrium payoff profile is enforceable

- Fix strategy profile s
- Define strategy s_i' of player i: for every h

$$s_i'(h) = \text{ best response to } s_{-i}(h)$$

- By definition of v_i, $u_i(s_{-i}(h), s_i'(h)) \geq v_i$ for every h

$\Rightarrow$ i's discounted average payoff to (s_{-i}, s_i') is $\geq v_i$

$\Rightarrow$ For s to be a Nash equilibrium of repeated game we need i's payoff $\geq v_i$

$\Rightarrow$ Every Nash equilibrium payoff profile is enforceable

Nash equilibrium payoffs

When players are very patient, set of Nash equilibrium payoff profiles essentially $=$ set of enforceable payoff profiles

Proposition (*Nash folk theorem*)

Let w be a strictly enforceable payoff profile of a strategic game G. For all $\varepsilon > 0$ there exists $\underline{\delta} < 1$ such that if $\delta > \underline{\delta}$ then the δ-discounted infinitely repeated game of G has a Nash equilibrium whose payoff profile w' satisfies $|w' - w| < \varepsilon$.

Idea of proof

If any player j deviates, other players hold j down to her minmax payoff v_j subsequently

Proof that every strictly enforceable payoff profile is Nash equilibrium payoff profile

- Let w be strictly enforceable payoff profile

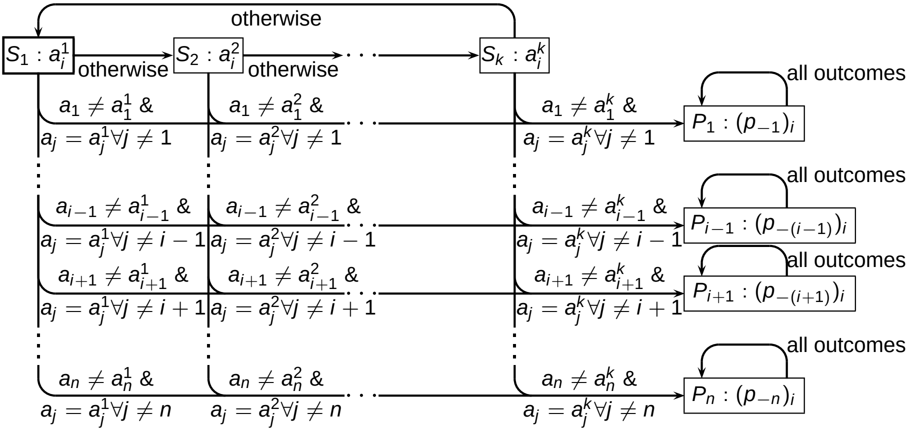
$\Rightarrow$ We can find outcome path $((a^1, a^2, \ldots, a^k), (a^1, a^2, \ldots, a^k), \ldots)$ of repeated game (where a^t is action profile of G for $t = 1, \ldots, k$) for which payoff profile is arbitrarily close to w

- For each player j, let p_{-j} be a list of actions of the other players that holds j's payoff to its minmax value, v_j:

$$p_{-j} \in \arg\min_{a_{-j} \in A_{-j}} \left(\max_{a_j \in A_j} u_j(a_{-j}, a_j) \right)$$

- Suppose each player i uses strategy that chooses her action in outcome path till first period in which a single player $j \neq i$ deviates, after which it chooses action $(p_{-j})_i$

Proof continued: i's strategy



Proof concluded

- The resulting strategy profile is a Nash equilibrium when players are sufficiently patient because any player j who deviates gets at most v_j in every period following her deviation
- Note that we do not need to worry about more than one player deviating in a period, because Nash equilibrium requires only that no *single* player can increase her payoff by deviating

Nash equilibrium payoffs of infinitely repeated *Prisoner's Dilemma*

Result implies that set of payoff pairs to Nash equilibria of infinitely repeated *Prisoner's Dilemma* is approximated, for δ close to 1, by shaded region in figure