ECO2030: Microeconomic Theory II, module 1

Lecture 10

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Repeated games

- *Same* set of players interact repeatedly

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- Extensive game with perfect information and simultaneous moves

Infinitely repeated games

Let $G = \langle N, (A_i), (\succsim_i) \rangle$ be strategic game; denote $A = \times_{i \in N} A_i$

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- $P(h) = N$ for all h
- $\succsim_i^*$ is a preference relation on A^∞ that extends $\succsim_i$ in the sense that if $(a^t) \in A^\infty$, $a \in A$, $a' \in A$, and $a \succsim_i a'$ then

$$(a^1, \ldots, a^{t-1}, a, a^{t+1}, \ldots) \succsim_i^* (a^1, \ldots, a^{t-1}, a', a^{t+1}, \ldots)$$

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Sequences differ only in tth component (a vs. a')

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Repeated games: Example

Suppose G is *Prisoner's Dilemma*

	C	D
C	$3, 3$	$0, 4$
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"[T]he strategies: [player 1] plays [C] 'til [player 2] plays [D], then [D] ever after, [player 2] plays [C] 'til [player 1] plays [D], then [D] ever after, are very nearly at equilibrium [in a 100-period repetition of the game] and in a game with an indeterminate stop point or an infinite game with interest on utility it is an equilibrium point."

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$$s_i^*(a^1,\ldots,a^{t-1}) = \begin{cases} & \text{if } a_j^\tau = C \text{ for } \tau = 1,\ldots,t-1 \\ & \end{cases}$$

where j is the other player

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  - payoffs $(3, 3)$ in every period

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- Strategy that chooses D in some period t after history in which P2 chose C in every previous period

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- Strategy that chooses D in some period t after history in which P2 chose C in every previous period
  - outcome in period t is (D, C), with payoffs $(4, 0)$

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  - in every subsequent period P2 chooses D
  - payoff to P1 in every subsequent period is at most 1

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  - outcome in period t is (D,C), with payoffs $(4,0)$
  - in every subsequent period P2 chooses D
  - payoff to P1 in every subsequent period is at most 1
  - best option for P1 is to choose D after period t

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$$\text{stick to } C \Rightarrow \text{payoffs } (3, 3, \ldots, 3, 3, 3, 3, \ldots)$$

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- If P1 is not too impatient, $(3, 3, \ldots)$ is better, so best response is strategy that chooses C after every history in which P2 chooses C in every period

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- If P1 is not too impatient, $(3,3,\dots)$ is better, so best response is strategy that chooses C after every history in which P2 chooses C in every period
- s_1^* is such a strategy
- Argument is symmetric for P2, so if players are sufficiently patient, (s_1^*, s_2^*) is a Nash equilibrium

Repeated games: Example

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Conclusion

If players sufficiently patient, strategy pair (s_1^*, s_2^*) is Nash equilibrium of infinitely repeated game, where $s_i^*(\varnothing) = C$ and

$$s_i^*(a^1, \ldots, a^{t-1}) = \begin{cases} C & \text{if } a_j^\tau = C \text{ for } \tau = 1, \ldots, t-1 \\ D & \text{otherwise} \end{cases}$$

where j is the other player

Outcome of this equilibrium is (C, C) in every period

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- How patient do the players have to be for the strategy pair (s_1^*, s_2^*) to be a Nash equilibrium?
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- What outcomes other than (C, C) in every period are supported?
- What about subgame perfect equilibria rather than Nash equilibria? Is it optimal for each player to punish the other player for deviating?
- What happens in games other than *Prisoner's Dilemma*?

Preferences in repeated games

Discounting Represented by discounted sum of one-shot payoffs: sequence $(a^1, a^2, \ldots)$ of outcomes has payoff

$$\sum_{t=1}^{\infty} \delta^{t-1} u_i(a^t)$$

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Overtaking Won't discuss

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G: payoffs (u_1, u_2)

	C	D
C	$7,8$	$1,11$
D	$9,-1$	$3,2$

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- Payoffs of P2: $v_2(a) = -1 + 3u_2(a)$ for all a

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	C	D
C	$7,8$	$1,11$
D	$9,-1$	$3,2$

G': payoffs (v_1, v_2)

- Payoffs of P1: $v_1(a) = 1 + 2u_1(a)$ for all a
- Payoffs of P2: $v_2(a) = -1 + 3u_2(a)$ for all a
- So *preferences* of player i in infinitely repeated game of G are same as preferences of player i in G', for $i = 1, 2$

Preferences in repeated games

- Will concentrate on preferences with discounting
- Two strategic games generate same preferences with discounting in repeated game $\Leftrightarrow$ each player's payoffs in one game are affine transformation of her payoffs in other game

Example

	C	D
C	$3,3$	$0,4$
D	$4,0$	$1,1$

G: payoffs (u_1, u_2)

	C	D
C	$3,3$	$0,7$
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- Payoffs of P1 are ordinally same in two games but *not* affine transforms of each other

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G': payoffs (u_1, u_2)

- Payoffs of P1 are ordinally same in two games but *not* affine transforms of each other
- Different preferences in repeated game: for δ close to one

$$((C,C),(C,C)) \succ_1 ((C,D),(D,C)) \text{ for left game}$$

$$((C,C),(C,C)) \prec_1 ((C,D),(D,C)) \text{ for right game}$$

Preferences with discounting

- Instead of working with discounted sum, sometimes convenient to work with discounted average

$$(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}u_i(a^t)$$

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- Sometimes refer to player's discounted average payoff simply as her payoff in repeated game

Nash equilibrium of *Prisoner's Dilemma*

Can now answer question: For the *Prisoner's Dilemma*

	C	D
C	$3, 3$	$0, 4$
D	$4, 0$	$1, 1$

how patient do the players have to be for the strategy pair (s_1^*, s_2^*) to be a Nash equilibrium, where

$$s_i^*(\varnothing) = C$$

$$s_i^*(a^1, \ldots, a^{t-1}) = \begin{cases} C & \text{if } a_j^\tau = C \text{ for } \tau = 1, \ldots, t-1 \\ D & \text{otherwise} \end{cases}$$

for $i = 1$, 2 and j is the other player?

Nash equilibrium of *Prisoner's Dilemma*

	C	D
C	$3, 3$	$0, 4$
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Suppose P2 uses s_2^*

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- P1 uses s_1^*

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  $\Rightarrow$ P1's payoff is $3 + 3\delta + 3\delta^2 + \cdots$

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- P1 deviates

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  - $\Rightarrow$ Best strategy of P1 that deviates in t chooses D in every period $\geq t + 1$
  - $\Rightarrow$ P1's payoff is $3 + 3\delta + \cdots + 3\delta^{t-2} + 4\delta^{t-1} + \delta^t + \delta^{t+1} + \cdots$

Nash equilibrium of *Prisoner's Dilemma*

- P1 has no profitable deviation if and only if

$$3+3\delta+\cdots+3\delta^{t-2}+4\delta^{t-1}+\delta^{t}+\delta^{t+1}+\cdots \leq 3+3\delta+3\delta^{2}+\cdots$$

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$$4+\frac{\delta}{1-\delta}\leq\frac{3}{1-\delta}\quad\Leftrightarrow\quad 4(1-\delta)+\delta\leq 3\quad\Leftrightarrow\quad \delta\geq\frac{1}{3}$$

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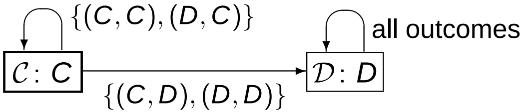
Conclusion: (s_1^*, s_2^*) is a Nash equilibrium if and only if $\delta \geq \frac{1}{3}$

Describing strategies

Can represent strategies compactly in figure:

s_1^*:



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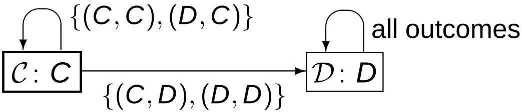

Viewed this way, the strategy is an automaton, consisting of

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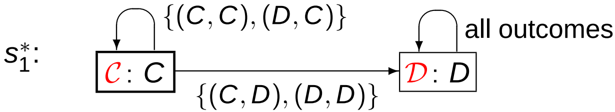

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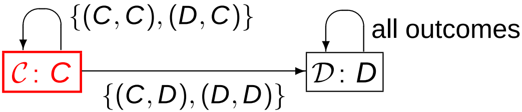

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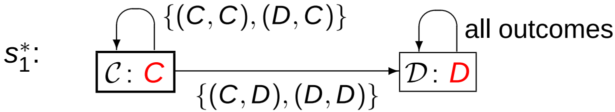

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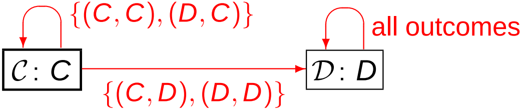

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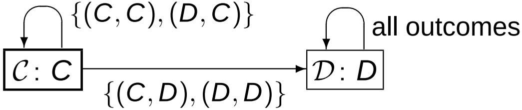

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Additional benefit of representing strategy in this way: measure of complexity of strategy is number of states

Describing strategies

Any automaton $\langle Q_i, q_i^0, f_i, \tau_i \rangle$ defines a strategy as follows:

- $s_i(\varnothing) = f_i(q_i^0)$

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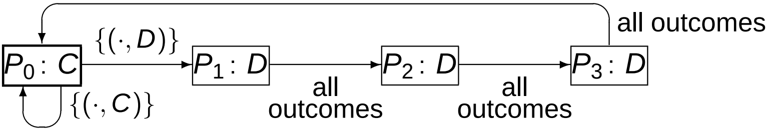
- and so on

Strategies: Examples

Three-period punishment



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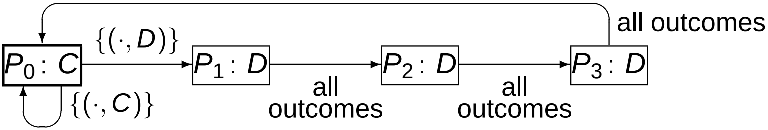

Tit-for-tat

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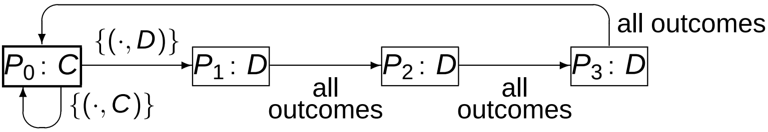

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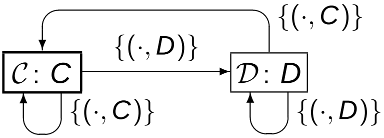

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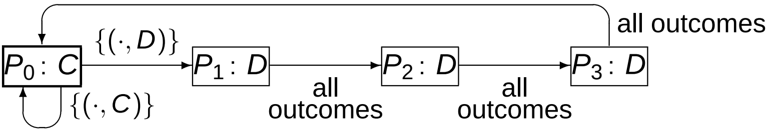

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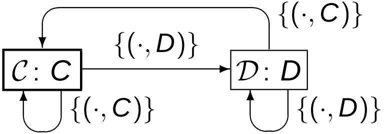

All-D

Strategies: Examples

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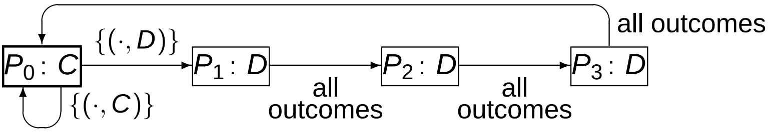

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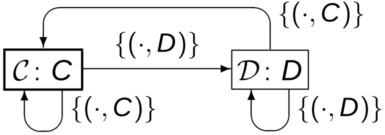

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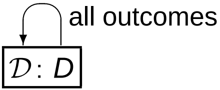
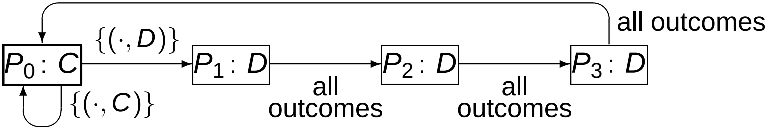


Nash equilibrium with limited punishment?

	C	D
C	$3,3$	$0,4$
D	$4,0$	$1,1$



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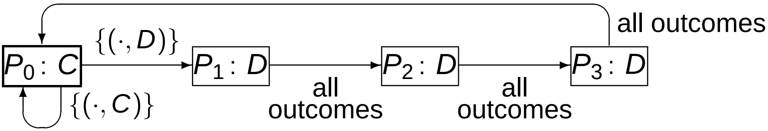

- Does infinitely repeated game have Nash equilibrium in which each player uses limited punishment strategy?

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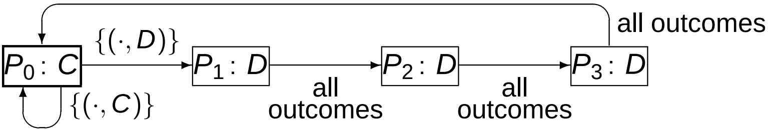

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- Suppose P2 uses limited punishment strategy with k periods of punishment

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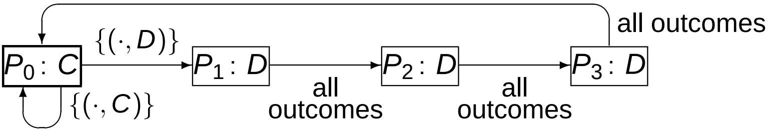

- Does infinitely repeated game have Nash equilibrium in which each player uses limited punishment strategy?
- Suppose P2 uses limited punishment strategy with k periods of punishment
- When is it optimal for P1 to use same strategy?

Nash equilibrium with limited punishment?

	C	D
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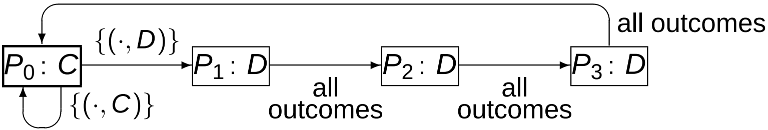

- If P1 uses same strategy, outcome is

Nash equilibrium with limited punishment?

	C	D
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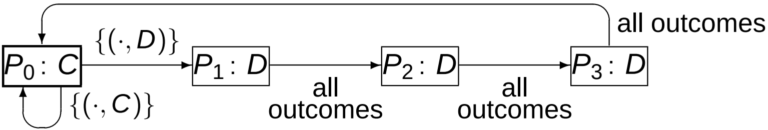
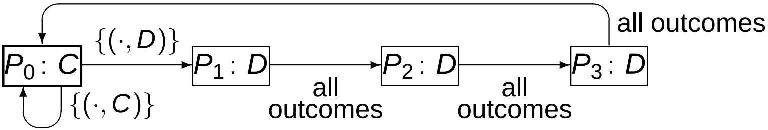


- If P1 uses same strategy, outcome is (C, C) in every period $\Rightarrow$ P1's payoff is 3 in every period

Nash equilibrium with limited punishment?

	C	D
C	3, 3	0, 4
D	4, 0	1, 1



- If P1 has profitable deviation, then deviation to D in period 1 that returns to C in period $k + 2$ is profitable

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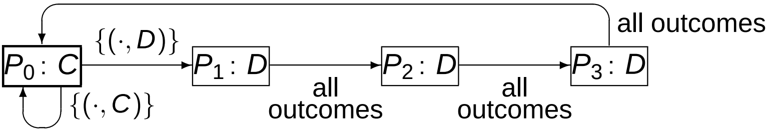

- If P1 has profitable deviation, then deviation to D in period 1 that returns to C in period $k+2$ is profitable
  Resulting outcomes and payoffs to P1:

(D, C)	4	
(D, D)	1	first period of punishment
(D, D)	1	
$\vdots$	$\vdots$	
(D, D)	1	last period of punishment
(C, C)	3	
(C, C)	3	

Nash equilibrium with limited punishment?

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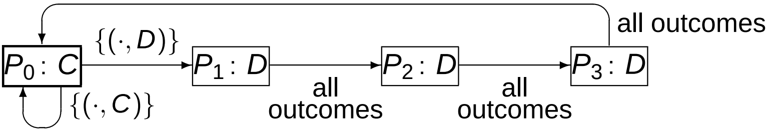

- For deviation not to be profitable, need

$$4 + \delta + \delta^2 + \cdots + \delta^k \leq 3 + 3\delta + 3\delta^2 + \cdots + 3\delta^k$$

Nash equilibrium with limited punishment?

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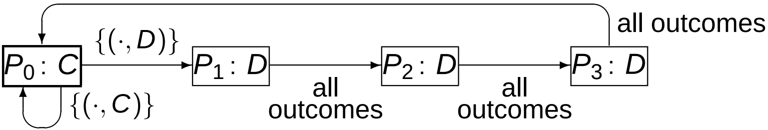

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$$3 + (1 + \delta + \delta^2 + \cdots + \delta^k) \leq 3(1 + \delta + \delta^2 + \cdots + \delta^k)$$

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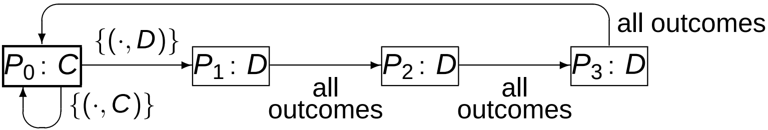

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$$3 \leq \frac{2(1 - \delta^{k+1})}{1 - \delta}$$

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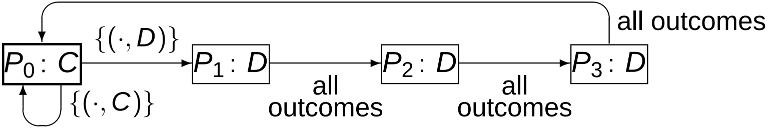

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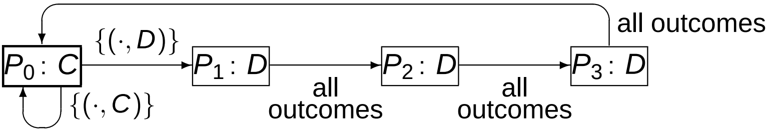

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$$1 - 3\delta + 2\delta^{k+1} \leq 0$$

$k = 1 \Rightarrow \delta \geq \frac{1}{2}$; $k \uparrow \Rightarrow$ cutoff value of $\delta \downarrow \frac{1}{3}$

Nash equilibrium with limited punishment?

Conclusion

- For any value of $k \geq 1$, strategy pair in which each player punishes other for k periods in event of deviation is Nash equilibrium of infinitely repeated game if δ is close enough to 1

Nash equilibrium with limited punishment?

Conclusion

- For any value of $k \geq 1$, strategy pair in which each player punishes other for k periods in event of deviation is Nash equilibrium of infinitely repeated game if δ is close enough to 1
- Larger $k \Rightarrow$ smaller lower bound on δ: mutually desirable outcome (C, C) is sustained by short punishment only if players are relatively patient

What payoffs can be achieved in a Nash equilibrium?

Feasible payoffs

First consider payoffs that are *possible* when players are patient (discounting with δ close to 1, or limit of means)

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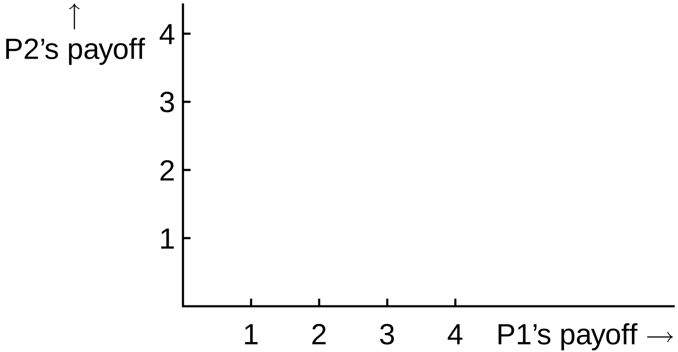

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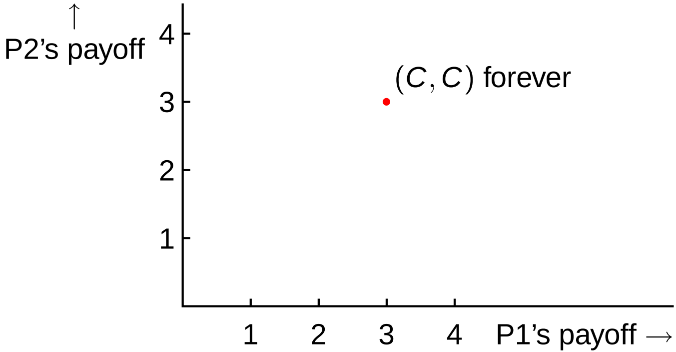

- Action pair (C, C) in every period $\Rightarrow$ (discounted average, and limit of means) payoffs $(3, 3)$

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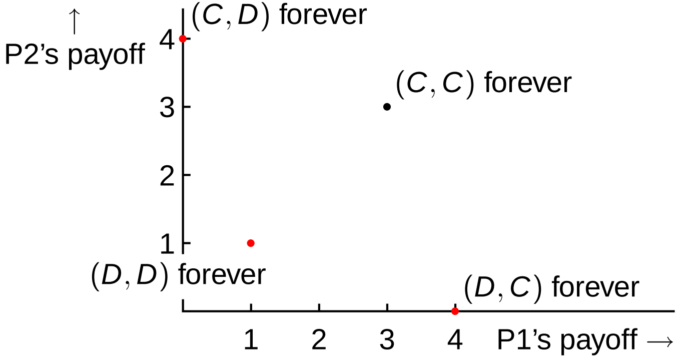

- Similarly for repetitions of other action pairs

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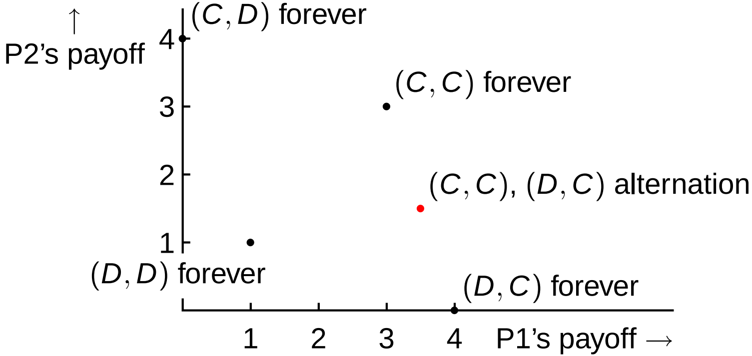

- Could alternate between (C, C) and (D, C) $\Rightarrow$ payoffs close to $(\frac{7}{2}, \frac{3}{2})$

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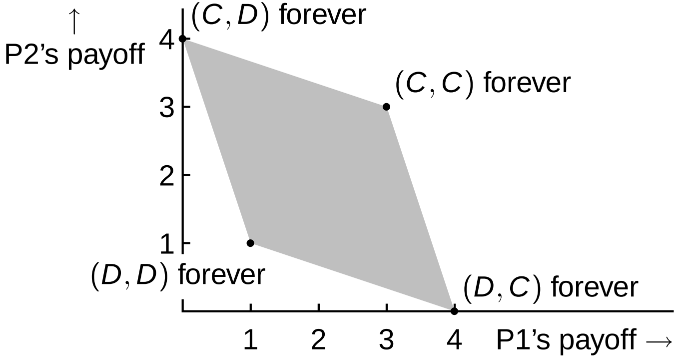
First consider payoffs that are *possible* when players are patient (discounting with δ close to 1, or limit of means)



- Similarly could cycle through any other sequence of outcomes $\Rightarrow$ average of payoffs to outcomes in sequence

What payoffs can be achieved in a Nash equilibrium?

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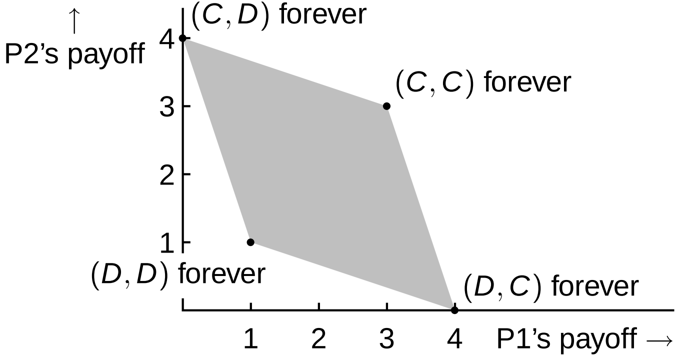
First consider payoffs that are *possible* when players are patient (discounting with δ close to 1, or limit of means)



- Can approximately achieve any linear combination of payoffs in stage game

Nash equilibrium payoffs

One Nash equilibrium

Pair (s^*, s^*) of punishment strategies is a Nash equilibrium of repeated game, yielding discounted average payoffs $(3, 3)$

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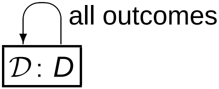
Another Nash equilibrium

Consider strategy $\hat{s}$ in which each player chooses D after every history:



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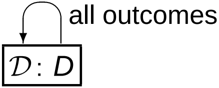
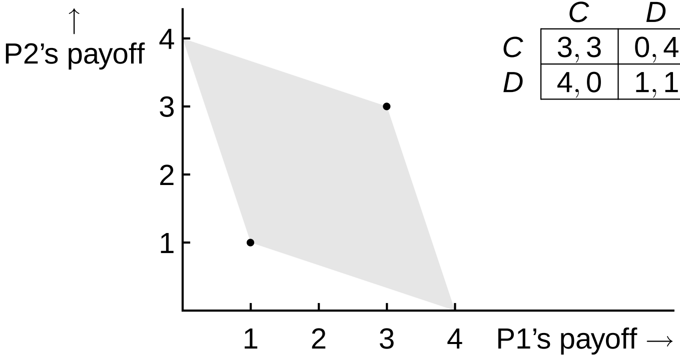


$(\hat{s}, \hat{s})$ is a Nash equilibrium of the repeated game, yielding payoffs $(1, 1)$

Nash equilibrium payoffs

These equilibria yield payoffs of $(3, 3)$ and $(1, 1)$. What other payoffs are possible in Nash equilibria?



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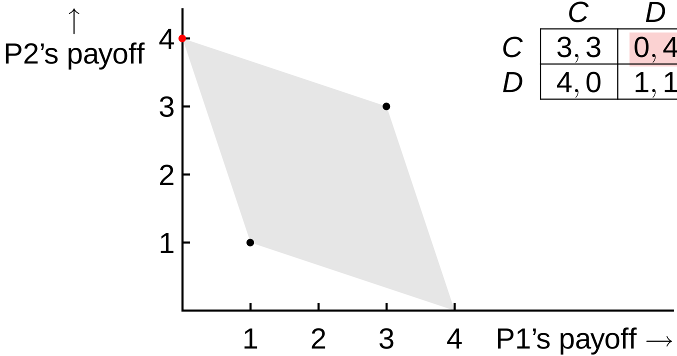

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- Are payoffs $(0, 4)$ possible in Nash equilibrium?

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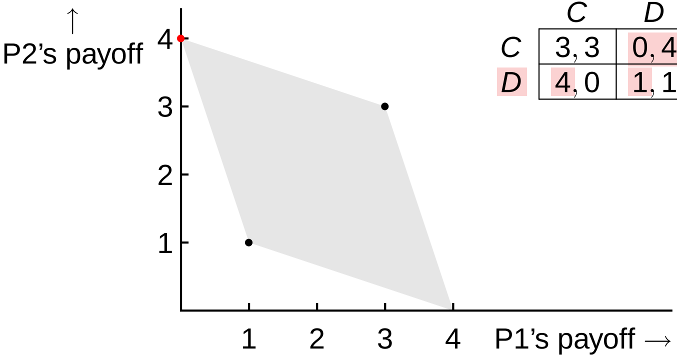

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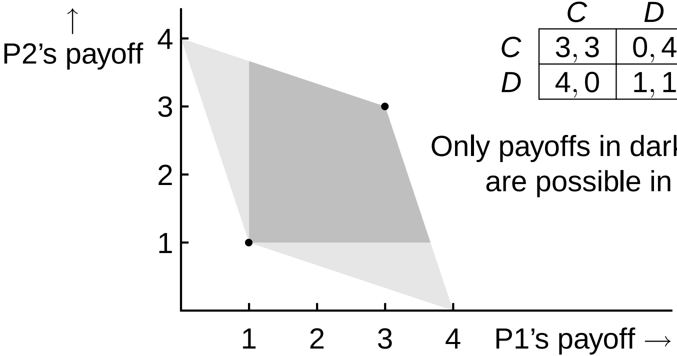

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Only payoffs in dark region are possible in NE

- Are payoffs $(0,4)$ possible in Nash equilibrium?
- No, because by choosing *D* after every history, P1 *guarantees* payoff of at least 1 in every period
- In any Nash equilibrium, payoff of each player is at least 1

Nash equilibrium payoffs

For general strategic game, the payoff player i can guarantee in any period is her minmax payoff

$$\max_{a_i \in A_i} u_i(a_{-i}, a_i)$$

Highest payoff possible for i given a_{-i}

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Example

	A	B	C
A	1, 1	0, 0	2, 3
B	0, 0	1, 2	1, 2
C	0, 2	2, 3	3, 1

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- $v_2 = 2$
- Note that pair of minmax actions (B, A) is not a Nash equilibrium of this game
- For *Prisoner's Dilemma*, $v_1 = v_2 = 1$, and pair of minmax actions, (C, C), *is* a Nash equilibrium

Nash equilibrium payoffs

- $w_i \geq v_i$ for all $i \in N \Rightarrow w$ is enforceable
- $w_i > v_i$ for all $i \in N \Rightarrow w$ is strictly enforceable

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For any strategic game G and any discount factor δ, every Nash equilibrium payoff profile of the δ-discounted infinitely repeated game of G is an enforceable payoff profile of G

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Idea of proof

Every player i can get at least v_i in every period by choosing an action in the period that best responds to the other players' actions

Proof that every Nash equilibrium payoff profile is enforceable

- Fix strategy profile s

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- Define strategy s_i' of player i: for every h

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Nash equilibrium payoffs

When players are very patient, set of Nash equilibrium payoff profiles essentially $=$ set of enforceable payoff profiles

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Proposition (*Nash folk theorem*)

Let w be a strictly enforceable payoff profile of a strategic game G. For all $\varepsilon > 0$ there exists $\underline{\delta} < 1$ such that if $\delta > \underline{\delta}$ then the δ-discounted infinitely repeated game of G has a Nash equilibrium whose payoff profile w' satisfies $|w' - w| < \varepsilon$.

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Idea of proof

If any player j deviates, other players hold j down to her minmax payoff v_j subsequently

Proof that every strictly enforceable payoff profile is Nash equilibrium payoff profile

- Let w be strictly enforceable payoff profile

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$\Rightarrow$ We can find outcome path $((a^1, a^2, \ldots, a^k), (a^1, a^2, \ldots, a^k), \ldots)$ of repeated game (where a^t is action profile of G for $t = 1, \ldots, k$) for which payoff profile is arbitrarily close to w

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- For each player j, let p_{-j} be a list of actions of the other players that holds j's payoff to its minmax value, v_j:

$$p_{-j} \in \arg\min_{a_{-j} \in A_{-j}} \left(\max_{a_j \in A_j} u_j(a_{-j}, a_j) \right)$$

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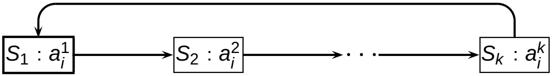
- Suppose each player i uses strategy that chooses her action in outcome path till first period in which a single player $j \neq i$ deviates, after which it chooses action $(p_{-j})_i$

Proof continued: i's strategy



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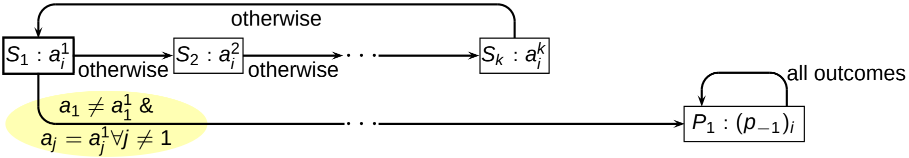

P1 alone deviates from a^1

Pi joins in punishing P1 forever

Proof continued: i's strategy

No player deviates from a^1 or
Pi alone deviates from a^1 or
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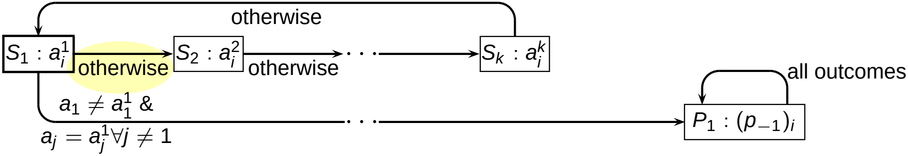
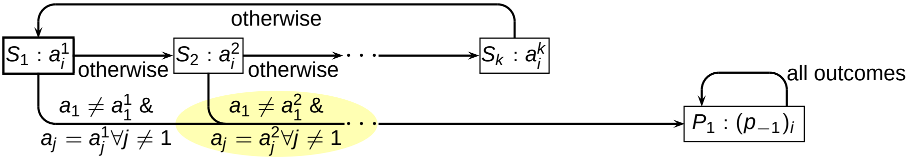


Proof continued: *i*'s strategy



P1 alone deviates from a^2

Pi joins in punishing P1 forever

Proof continued: i's strategy

No player deviates from a^2 or
Pi alone deviates from a^2 or
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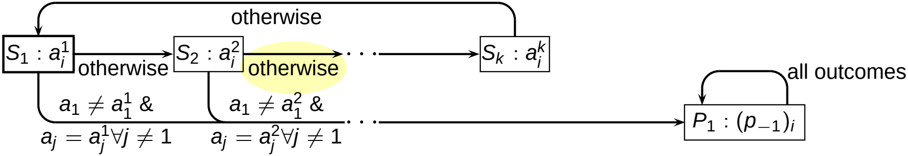
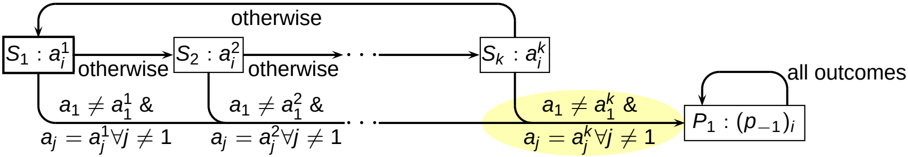
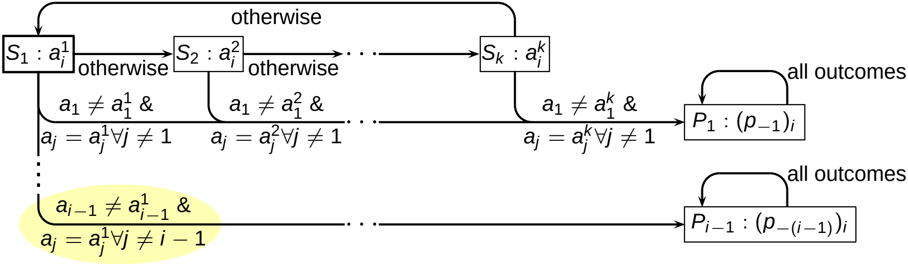


Proof continued: *i*'s strategy



P1 alone deviates from a^k

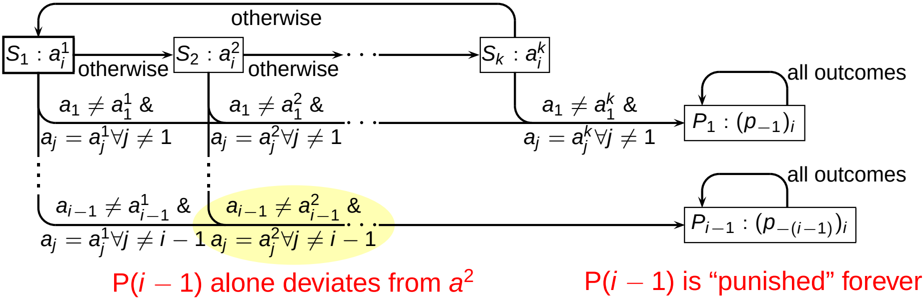
Proof continued: i's strategy



P($i-1$) alone deviates from a^1

P($i-1$) is "punished" forever

Proof continued: i's strategy



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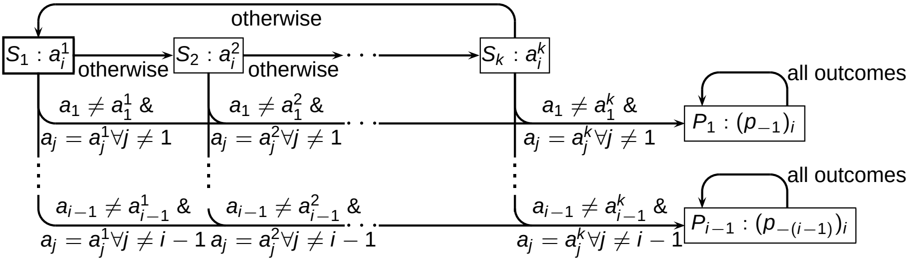
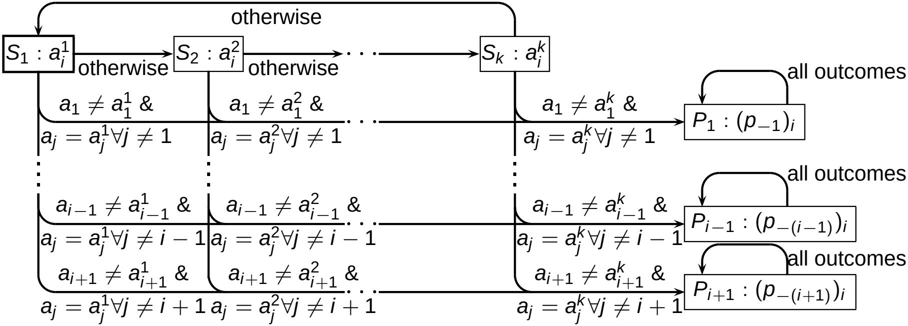
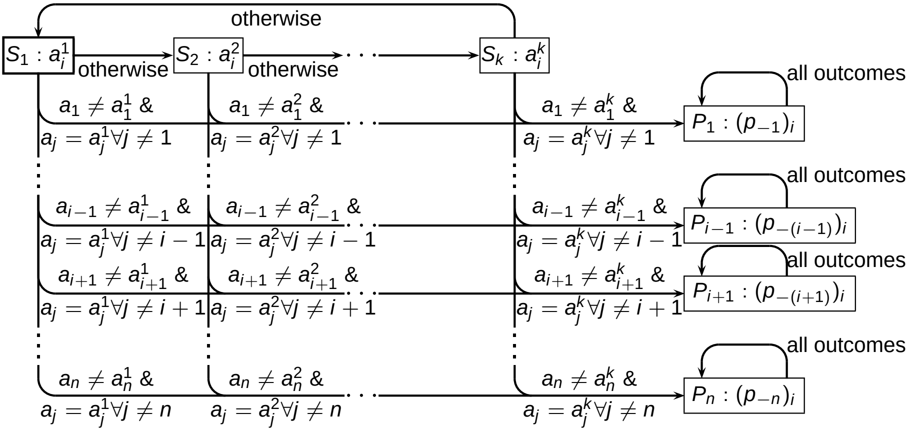


Proof continued: i's strategy



Proof continued: i's strategy



Proof concluded

- The resulting strategy profile is a Nash equilibrium when players are sufficiently patient because any player j who deviates gets at most v_j in every period following her deviation

Proof concluded

- The resulting strategy profile is a Nash equilibrium when players are sufficiently patient because any player j who deviates gets at most v_j in every period following her deviation
- Note that we do not need to worry about more than one player deviating in a period, because Nash equilibrium requires only that no *single* player can increase her payoff by deviating

Nash equilibrium payoffs of infinitely repeated *Prisoner's Dilemma*

Result implies that set of payoff pairs to Nash equilibria of infinitely repeated *Prisoner's Dilemma* is approximated, for δ close to 1, by shaded region in figure