ECO2030: Microeconomic Theory II, module 1

Lecture 9

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Nash's axiomatic model of bargaining

Axiomatic approach

- *Bargaining problem* is specified by
  - set of possible agreements
  - outcome in case of disagreement
  - players' preferences over possible outcomes
- *Bargaining solution* associates outcome with every bargaining problem
- Specify properties of bargaining solution that seem reasonable and find all solutions that satisfy these properties
- Chapter 15 of book, but here I take standard approach, as in Exercise 309.1 or Chapter 3 of *Bargaining and Markets*

Nash's axiomatic model of bargaining

- Two individuals
- X: set of possible agreements
- D: outcome in case of disagreement
- Players have preferences over $X \cup \{D\}$

Nash's axiomatic model of bargaining

- Bargaining seems to entail risk, so specify players' preferences over *lotteries* over $X \cup \{D\}$
- Assume preferences satisfy vNM axioms, and hence are represented by expected values of Bernoulli payoffs
- u_i: player i's Bernoulli payoff function on $X \cup \{D\}$
- Let

$$U = \{(u_1(x), u_2(x)) : x \in X\}$$
$$d = (u_1(D), u_2(D))$$

- Subsequently will take (U, d) as primitive, rather than (X, D)

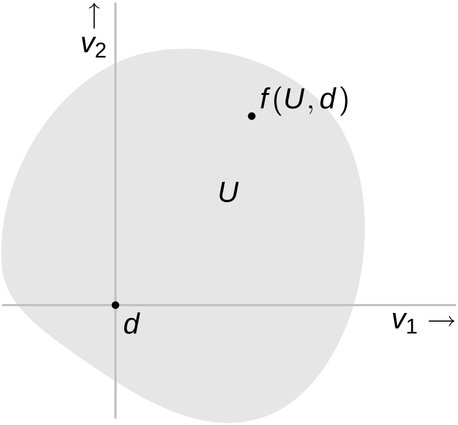
Nash's axiomatic model of bargaining

Definition

A bargaining problem is a pair (U, d), where $U \subset \mathbb{R}^2$ and $d \in U$ (disagreement is a possible outcome), such that

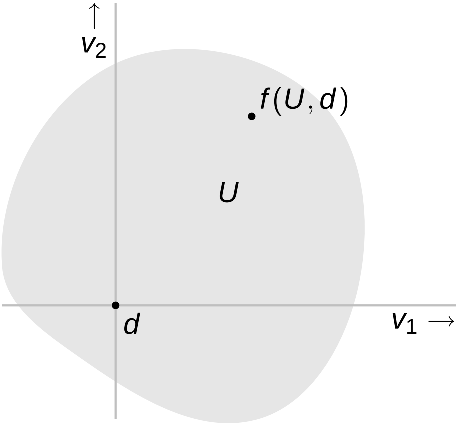
- there exists $(v_1, v_2) \in U$ such that $v_1 > d_1$ and $v_2 > d_2$ (some agreement is better for both players than disagreement)
- U is convex, bounded, and closed

Nash's axiomatic model of bargaining



A bargaining problem (U, d)

Nash's axiomatic model of bargaining



Definition

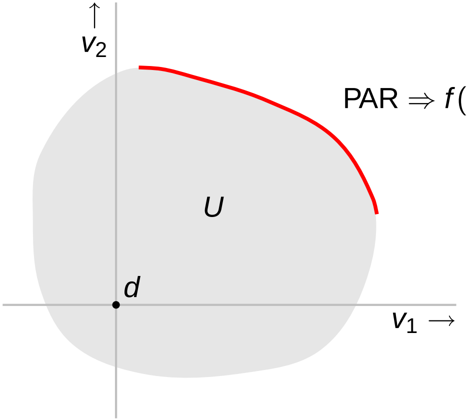
A bargaining solution is a function f that associates with every bargaining problem (U,d) a member $f(U,d)$ of U

Nash's axiomatic model of bargaining: Axioms

What conditions should a bargaining solution satisfy?

Pareto efficiency (PAR)

If $v \in U$, $v' \in U$, & $v_i > v'_i$ for $i = 1, 2$, then $f(U, d) \neq v'$



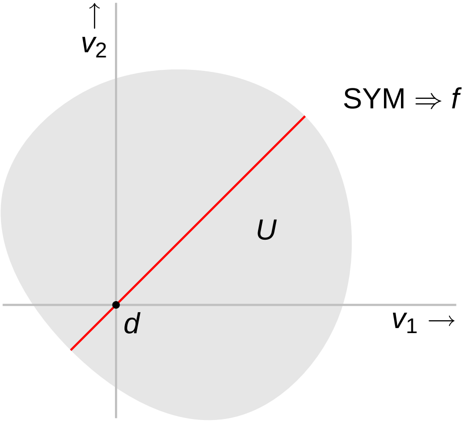
PAR $\Rightarrow f(U, d)$ on red line

Nash's axiomatic model of bargaining: Axioms

Symmetry (SYM)

If $(v_1, v_2) \in U \Leftrightarrow (v_2, v_1) \in U$ and $d_1 = d_2$, then

$$f_1(U,d) = f_2(U,d)$$

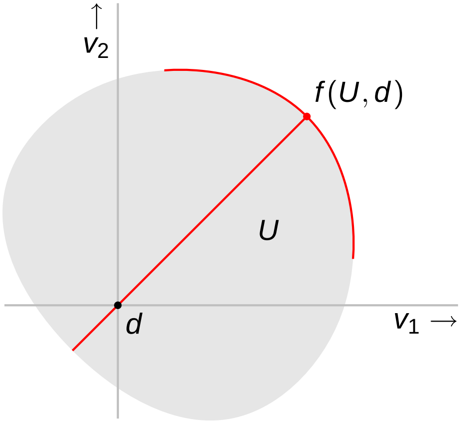


SYM $\Rightarrow f(U,d)$ on red line

Nash's axiomatic model of bargaining: Axioms

Symmetry and efficiency

- SYM directly restricts solution *only* for symmetric problems
- If U is symmetric and $d_1 = d_2$ then PAR and SYM $\Rightarrow$ $f(U, d)$ is point v on Pareto frontier of U for which $v_1 = v_2$



Nash's axiomatic model of bargaining: Axioms

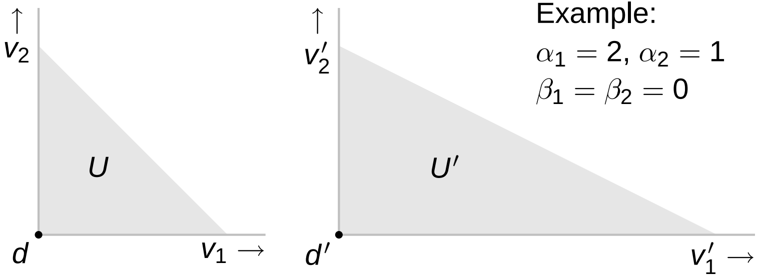
- Outcome of bargaining should depend on individuals' *preferences*, not the representation of these preferences
- Bargaining problem (U, d) entails same *preferences* as bargaining problem (U', d') in which

$$U' = \{(\alpha_1 v_1 + \beta_1, \alpha_2 v_2 + \beta_2) : (v_1, v_2) \in U\}$$
$$d' = (\alpha_1 d_1 + \beta_1, \alpha_2 d_2 + \beta_2)$$

for some $\alpha_i > 0$ and β_i, $i = 1, 2$

Nash's axiomatic model of bargaining: Axioms

$$U' = \{(\alpha_1 v_1 + \beta_1, \alpha_2 v_2 + \beta_2) : (v_1, v_2) \in U\}$$
$$d' = (\alpha_1 d_1 + \beta_1, \alpha_2 d_2 + \beta_2)$$



Example:
$\alpha_1 = 2$, $\alpha_2 = 1$
$\beta_1 = \beta_2 = 0$

Players' preferences are the *same* in (U, d) and (U', d'); only representations of preferences differ

Nash's axiomatic model of bargaining: Axioms

Outcome should be independent of payoff representations $\Rightarrow$ solution should co-vary with payoff representation

Covariance with positive affine transformations (INV)

Let $\alpha_i > 0$ and β_i for $i = 1, 2$ be numbers, let

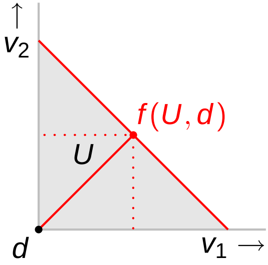
$$U' = \{(\alpha_1 v_1 + \beta_1, \alpha_2 v_2 + \beta_2) : (v_1, v_2) \in U\}$$

and let $d' = (\alpha_1 d_1 + \beta_1, \alpha_2 d_2 + \beta_2)$. Then

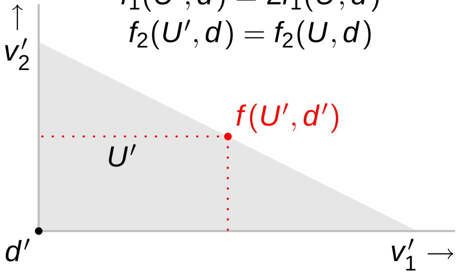
$$f_i(U', d') = \alpha_i f_i(U, d) + \beta_i \text{ for } i = 1, 2.$$

Nash's axiomatic model of bargaining: Axioms

PAR & SYM $\Rightarrow$



$$U' = \{(2v_1, v_2) : (v_1, v_2) \in U\}$$
and $d' = (2d_1, d_2)$,
so PAR & SYM & INV $\Rightarrow$
$$f_1(U', d) = 2f_1(U, d)$$
$$f_2(U', d) = f_2(U, d)$$

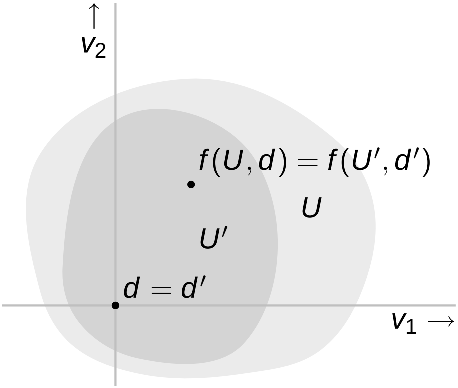


Nash's axiomatic model of bargaining: Axioms

- INV extends implications of PAR and SYM to affine transformations of symmetric problems
- To extend implications to other problems, new axiom

Independence of irrelevant alternatives (IIA)

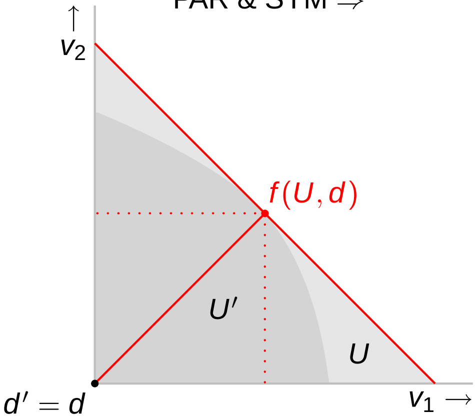
If $U' \subseteq U$, $d' = d$, and $U' \ni f(U,d)$ then $f(U',d') = f(U,d)$



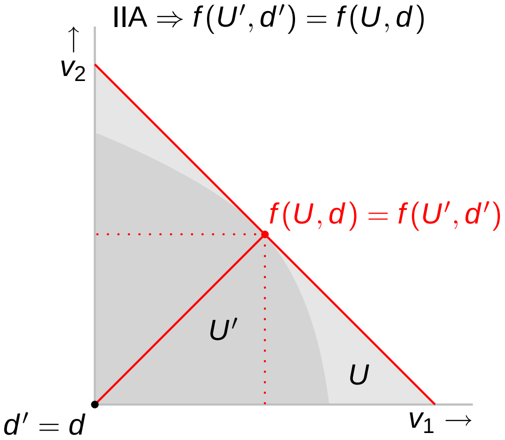
Idea: if $f(U,d) \in U'$ then members of $U \setminus U'$ are irrelevant

Nash's axiomatic model of bargaining: Axioms

PAR & SYM $\Rightarrow$



Nash's axiomatic model of bargaining: Axioms



Nash Bargaining Solution

Proposition

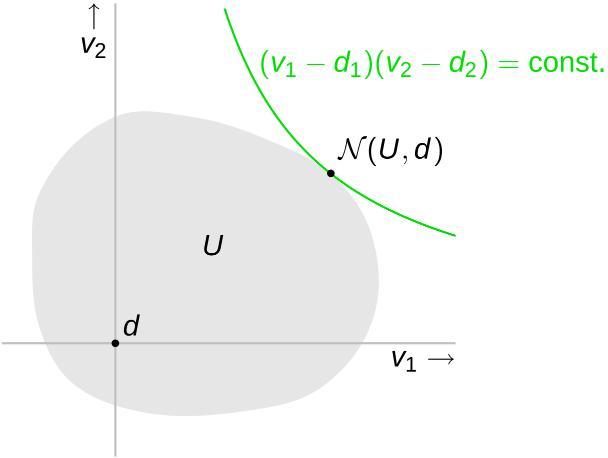
A unique bargaining solution satisfies the axioms INV, SYM, IIA, and PAR. This solution is given by

$$\mathcal{N}(U,d) = \arg\max_{(v_1,v_2)} (v_1 - d_1)(v_2 - d_2)$$

$$\text{s.t. } (v_1,v_2) \in U \text{ and } (v_1,v_2) \geq (d_1,d_2).$$

$\mathcal{N}(U,d)$ is the Nash solution of the bargaining problem (U,d)

Nash Bargaining Solution



Nash Bargaining Solution: Proof of Proposition

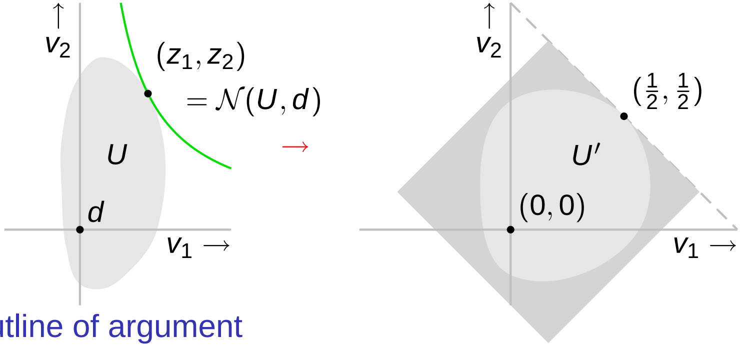
- $\mathcal{N}$ satisfies axioms: exercise (for INV, see Problem Set)
- Let (U, d) be any bargaining problem
- Need to show axioms $\Rightarrow$ solution of (U, d) is $\mathcal{N}(U, d)$
- Denote $z = \mathcal{N}(U, d)$ and let

$$\alpha_i = \frac{1}{2(z_i - d_i)} \quad \text{and} \quad \beta_i = \frac{-d_i}{2(z_i - d_i)} \quad \text{for } i = 1, 2$$

- Note that $\alpha_i z_i + \beta_i = \frac{1}{2}$ and $\alpha_i d_i + \beta_i = 0$ for $i = 1, 2$
- Define

$$U' = \{(\alpha_1 y_1 + \beta_1, \alpha_2 y_2 + \beta_2) : (y_1, y_2) \in U\}$$
$$d_i' = \alpha_i d_i + \beta_i = 0 \quad \text{for } i = 1, 2$$

Nash Bargaining Solution: Proof of Proposition



Outline of argument

1. Axioms $\Rightarrow$ solution of (U, d) is $\mathcal{N}(U, d)$
   iff axioms $\Rightarrow$ solution of (U', d') is $\mathcal{N}(U', d')$
2. $\mathcal{N}(U', d') = (\frac{1}{2}, \frac{1}{2})$
3. Show that U' lies below the line $v_1 + v_2 = 1$
   $\Rightarrow U'$ is subset of symmetric set that includes $(\frac{1}{2}, \frac{1}{2})$
4. Axioms $\Rightarrow$ solution of (U', d') is $(\frac{1}{2}, \frac{1}{2})$

Nash Bargaining Solution: Proof of Proposition

Step 1: Axioms $\Rightarrow$ solution of (U, d) is $\mathcal{N}(U, d)$ iff axioms $\Rightarrow$ solution of (U', d') is $\mathcal{N}(U', d')$

- Let f be bargaining solution that satisfies axioms
- f satisfies INV $\Rightarrow f_i(U', d') = \alpha_i f_i(U, d) + \beta_i$ for $i = 1, 2$
- $\mathcal{N}$ satisfies INV [Problem Set 9] $\Rightarrow$

$$\mathcal{N}_i(U', d') = \alpha_i \mathcal{N}_i(U, d) + \beta_i \text{ for } i = 1, 2$$

- Thus

$$\begin{aligned} f_i(U', d') = \mathcal{N}_i(U', d') \text{ for } i = 1, 2 \\ \Leftrightarrow \quad f_i(U, d) = \mathcal{N}_i(U, d) \text{ for } i = 1, 2. \end{aligned}$$

- That is, $f(U, d) = \mathcal{N}(U, d) \Leftrightarrow f(U', d') = \mathcal{N}(U', d')$

Nash Bargaining Solution: Proof of Proposition

Step 2: $\mathcal{N}(U', d') = (\frac{1}{2}, \frac{1}{2})$

- $\mathcal{N}$ satisfies INV $\Rightarrow \mathcal{N}_i(U', d') = \alpha_i \mathcal{N}_i(U, d) + \beta_i$ for $i = 1, 2$
- $\alpha_i \mathcal{N}_i(U, d) + \beta_i = \frac{1}{2}$ by definition of α_i and β_i, for $i = 1, 2$
- So $\mathcal{N}(U', d') = (\frac{1}{2}, \frac{1}{2})$

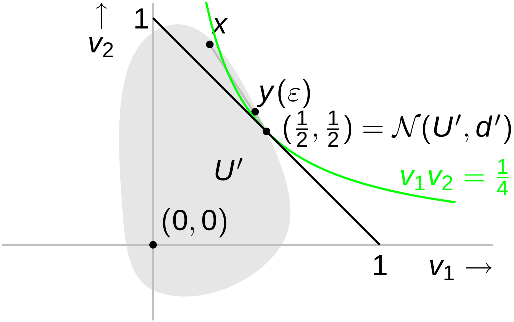
Nash Bargaining Solution: Proof of Proposition

Step 3: U' contains no point (x_1, x_2) with $x_1 + x_2 > 1$

- Suppose $(x_1, x_2) \in U'$ with $x_1 + x_2 > 1$
- For any $\varepsilon > 0$ let $y_i(\varepsilon) = (1-\varepsilon)\cdot\frac{1}{2} + \varepsilon x_i$ for $i = 1, 2$

$\Rightarrow$ $y_1(\varepsilon)y_2(\varepsilon) = \frac{1}{4} + \frac{1}{2}\varepsilon(x_1 + x_2 - 1) + \varepsilon^2[\frac{1}{4} - \frac{1}{2}(x_1 + x_2) + x_1x_2]$

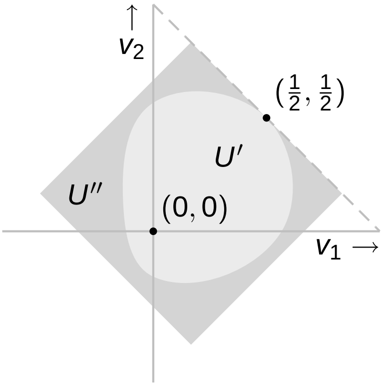
$\Rightarrow$ $y_1(\varepsilon)y_2(\varepsilon) > \frac{1}{4}$ for small ε, contradicting $\mathcal{N}(U', d') = (\frac{1}{2}, \frac{1}{2})$



Nash Bargaining Solution: Proof of Proposition

Step 3 continued: U' is subset of symmetric set that includes $(\frac{1}{2}, \frac{1}{2})$

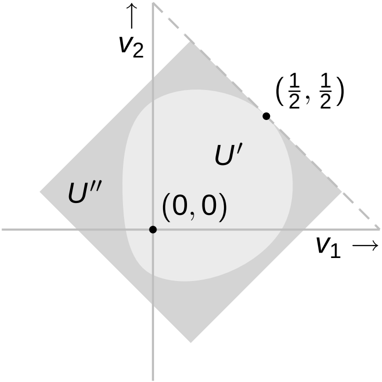
- Given U' contains no point (x_1, x_2) with $x_1 + x_2 = 1$, we can find symmetric rectangle U'' enclosing U' with Pareto surface intersecting $x_1 + x_2 = 1$



Nash Bargaining Solution: Proof of Proposition

Step 4: Axioms $\Rightarrow$ solution of (U', d') is $(\frac{1}{2}, \frac{1}{2})$

- By SYM and PAR we have $f(U'', d') = (\frac{1}{2}, \frac{1}{2})$
- By IIA we have $f(U', d') = (\frac{1}{2}, \frac{1}{2})$, completing the proof



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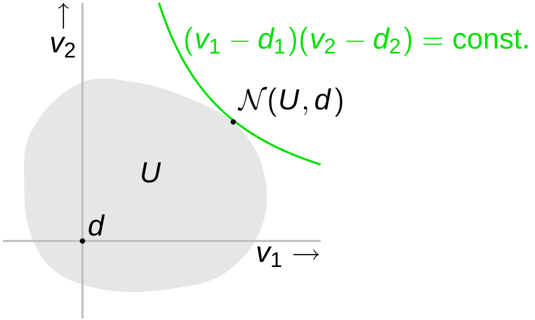


Relation between strategic and axiomatic models

- Consider bargaining game of alternating offers with risk of breakdown
- Let $U = \{(u_1(x_1), u_2(x_2)) : x_1 + x_2 \leq 1\}$
- If each player i is risk averse (u_i is concave), then
  - U is convex
  - U is bounded
  - U contains (b_1, b_2) (pair of breakdown payoffs)
  - U contains a pair of payoffs (v_1, v_2) such that $v_i > b_i$ for each player i
- Thus (U, b) is a bargaining problem

Relation between strategic and axiomatic models

In bargaining game of alternating offers with risk of breakdown, SPE entails proposals $(\hat{x}_1(\alpha), \hat{x}_2(\alpha))$ and $(\hat{y}_1(\alpha), \hat{y}_2(\alpha))$ such that

$$u_1(\hat{y}_1(\alpha)) = (1-\alpha)u_1(\hat{x}_1(\alpha)) + \alpha b_1$$
$$u_2(\hat{x}_2(\alpha)) = (1-\alpha)u_2(\hat{y}_2(\alpha)) + \alpha b_2$$

or

$$u_1(\hat{y}_1(\alpha)) - b_1 = (1-\alpha)\left(u_1(\hat{x}_1(\alpha)) - b_1\right)$$
$$u_2(\hat{x}_2(\alpha)) - b_2 = (1-\alpha)\left(u_2(\hat{y}_2(\alpha)) - b_2\right)$$

so that

$$\left(u_1(\hat{x}_1(\alpha)) - b_1\right)\left(u_2(\hat{x}_2(\alpha)) - b_2\right) = \left(u_1(\hat{y}_1(\alpha)) - b_1\right)\left(u_2(\hat{y}_2(\alpha)) - b_2\right)$$

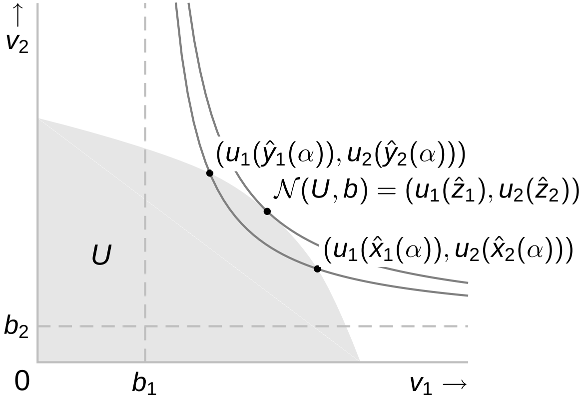
Relation between strategic and axiomatic models

$$(u_1(\hat{x}_1(\alpha)) - b_1)\,(u_2(\hat{x}_2(\alpha)) - b_2) = (u_1(\hat{y}_1(\alpha)) - b_1)\,(u_2(\hat{y}_2(\alpha)) - b_2)$$

$$\Leftrightarrow$$

$(u_1(\hat{y}_1(\alpha)), u_2(\hat{y}_2(\alpha)))$ and $(u_1(\hat{x}_1(\alpha)), u_2(\hat{x}_2(\alpha)))$

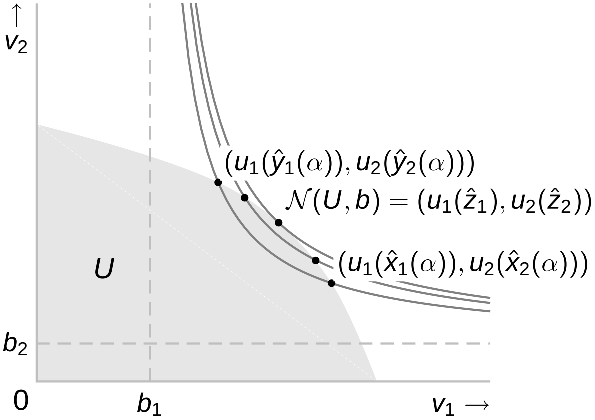
lie on same rectangular hyperbola relative to axes through (b_1, b_2)



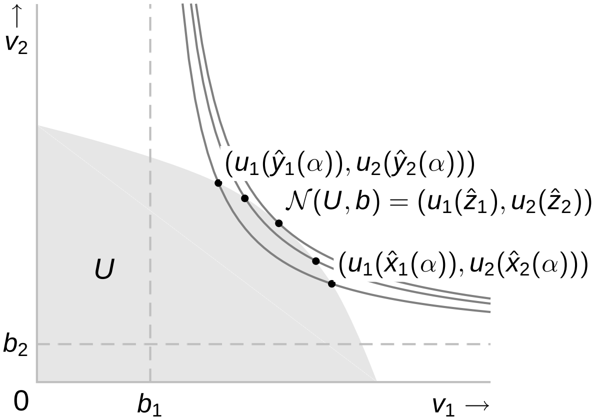
Relation between strategic and axiomatic models

$$u_1(\hat{y}_1(\alpha)) = (1-\alpha)u_1(\hat{x}_1(\alpha)) + \alpha b_1$$
$$u_2(\hat{x}_2(\alpha)) = (1-\alpha)u_2(\hat{y}_2(\alpha)) + \alpha b_2$$

$$\Rightarrow \quad \lim_{\alpha \to 0} (\hat{x}(\alpha) - \hat{y}(\alpha)) = (0,0)$$



Relation between strategic and axiomatic models



- Thus agreement $\hat{z}$ to which both $\hat{x}(\alpha)$ and $\hat{y}(\alpha)$ converge as $\alpha \to 0$ is Nash solution of (U, b)

Relation between strategic and axiomatic models

Proposition

The SPE outcome of the variant of the bargaining game of alternating offers with risk of breakdown (and discount factors of 1 for each player) converges to the Nash bargaining solution of the associated bargaining problem as the probability of breakdown converges to 0.

Relation between strategic and axiomatic models

- Bargaining game of alternating offers and Nash's bargaining solution complement each other
- Bargaining game of alternating offers assumes specific bargaining procedure; Nash bargaining model does not
- Result sheds light on disagreement payoffs in Nash bargaining model: should be breakdown payoffs—and not, for example, payoffs players receive when they *choose* to leave the bargaining table

Outside options vs. exogenous breakdown

- When applying bargaining model need to specify game appropriately—with either exogenous risk of breakdown or outside options
- Examples:
  - Two people negotiating split of proceeds of invention when risk of being scooped $\Rightarrow$ risk of breakdown
  - Buyer and seller, where buyer can choose to approach another seller $\Rightarrow$ outside option
- Disagreement point in Nash's model should be payoff in event of exogenous breakdown, *not* outside option payoff