ECO2030: Microeconomic Theory II, module 1

Lecture 9

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2018.11.27

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- *Bargaining problem* is specified by

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- *Bargaining problem* is specified by
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Nash's axiomatic model of bargaining

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Nash's axiomatic model of bargaining

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Nash's axiomatic model of bargaining

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  - set of possible agreements
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- Specify properties of bargaining solution that seem reasonable and find all solutions that satisfy these properties
- Chapter 15 of book, but here I take standard approach, as in Exercise 309.1 or Chapter 3 of *Bargaining and Markets*

Nash's axiomatic model of bargaining

- Two individuals

Nash's axiomatic model of bargaining

- Two individuals
- X: set of possible agreements

Nash's axiomatic model of bargaining

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Nash's axiomatic model of bargaining

- Two individuals
- X: set of possible agreements
- D: outcome in case of disagreement
- Players have preferences over $X \cup \{D\}$

Nash's axiomatic model of bargaining

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Utility possibility set

$$U = \{(u_1(x), u_2(x)) : x \in X\}$$
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- Subsequently will take (U, d) as primitive, rather than (X, D)

Nash's axiomatic model of bargaining

Definition

A bargaining problem is a pair (U, d), where $U \subset \mathbb{R}^2$ and $d \in U$ (disagreement is a possible outcome), such that

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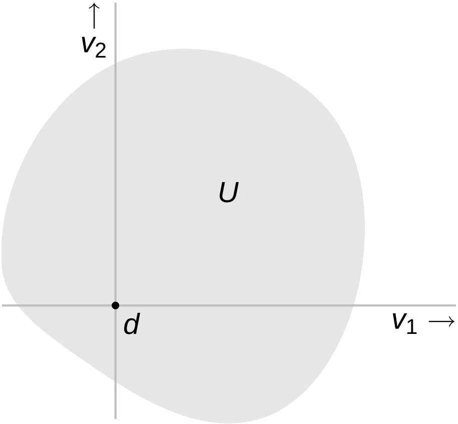
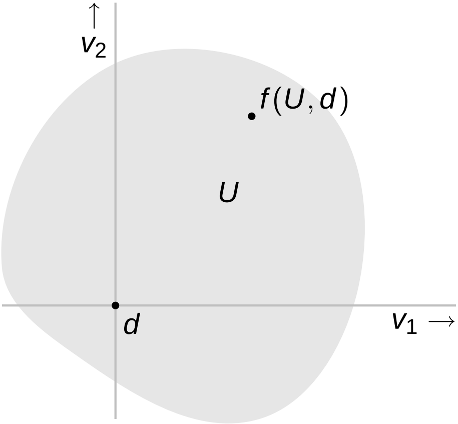
- there exists $(v_1, v_2) \in U$ such that $v_1 > d_1$ and $v_2 > d_2$ (some agreement is better for both players than disagreement)
- U is convex, bounded, and closed

Nash's axiomatic model of bargaining



A bargaining problem (U, d)

Nash's axiomatic model of bargaining



Definition

A bargaining solution is a function f that associates with every bargaining problem (U, d) a member $f(U, d)$ of U

Nash’s axiomatic model of bargaining: Axioms

What conditions should a bargaining solution satisfy?

Nash’s axiomatic model of bargaining: Axioms

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Pareto efficiency (PAR)

Nash's axiomatic model of bargaining: Axioms

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If $v \in U$, $v' \in U$, & $v_i > v'_i$ for $i = 1, 2$, then $f(U, d) \neq v'$

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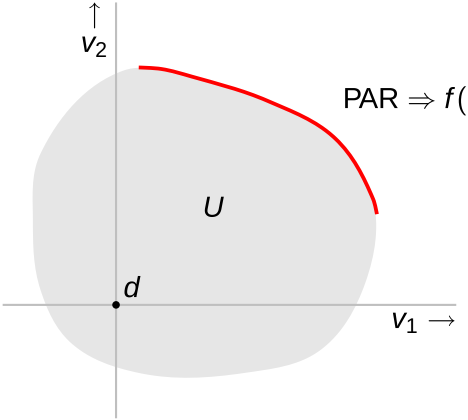

PAR $\Rightarrow f(U, d)$ on red line

Nash’s axiomatic model of bargaining: Axioms

Symmetry (SYM)

Nash's axiomatic model of bargaining: Axioms

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If $(v_1, v_2) \in U \Leftrightarrow (v_2, v_1) \in U$ and $d_1 = d_2$

Nash's axiomatic model of bargaining: Axioms

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If $(v_1, v_2) \in U \Leftrightarrow (v_2, v_1) \in U$ and $d_1 = d_2$, then

$$f_1(U, d) = f_2(U, d)$$

Nash's axiomatic model of bargaining: Axioms

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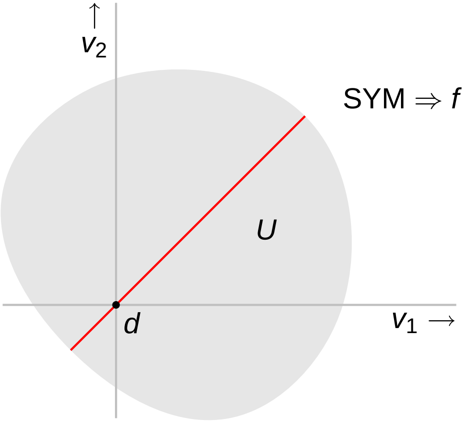


SYM $\Rightarrow f(U, d)$ on red line

Nash's axiomatic model of bargaining: Axioms

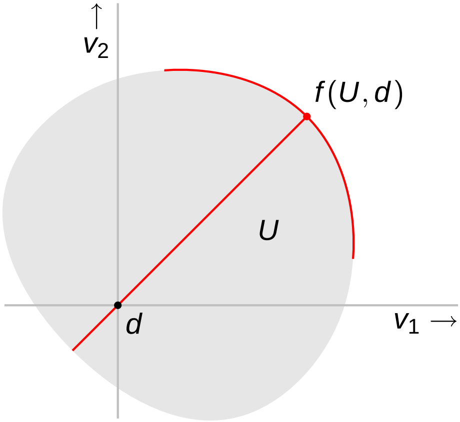
Symmetry and efficiency

- SYM directly restricts solution *only* for symmetric problems

Nash's axiomatic model of bargaining: Axioms

Symmetry and efficiency

- SYM directly restricts solution *only* for symmetric problems
- If U is symmetric and $d_1 = d_2$ then PAR and SYM $\Rightarrow$ $f(U, d)$ is point v on Pareto frontier of U for which $v_1 = v_2$



Nash's axiomatic model of bargaining: Axioms

- Outcome of bargaining should depend on individuals' *preferences*, not the representation of these preferences

Nash's axiomatic model of bargaining: Axioms

- Outcome of bargaining should depend on individuals' *preferences*, not the representation of these preferences
- Bargaining problem (U, d) entails same *preferences* as bargaining problem (U', d') in which

$$U' = \{(\alpha_1 v_1 + \beta_1, \alpha_2 v_2 + \beta_2) : (v_1, v_2) \in U\}$$
$$d' = (\alpha_1 d_1 + \beta_1, \alpha_2 d_2 + \beta_2)$$

for some $\alpha_i > 0$ and β_i, $i = 1, 2$

Nash's axiomatic model of bargaining: Axioms

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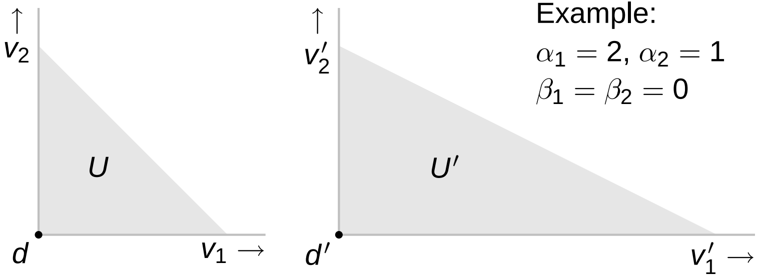

Example:
$\alpha_1 = 2$, $\alpha_2 = 1$
$\beta_1 = \beta_2 = 0$

Players' preferences are the *same* in (U, d) and (U', d'); only representations of preferences differ

Nash's axiomatic model of bargaining: Axioms

Outcome should be independent of payoff representations $\Rightarrow$ solution should co-vary with payoff representation

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Covariance with positive affine transformations (INV)

Let $\alpha_i > 0$ and β_i for $i = 1, 2$ be numbers, let

$$U' = \{(\alpha_1 v_1 + \beta_1, \alpha_2 v_2 + \beta_2) : (v_1, v_2) \in U\}$$

and let $d' = (\alpha_1 d_1 + \beta_1, \alpha_2 d_2 + \beta_2)$.

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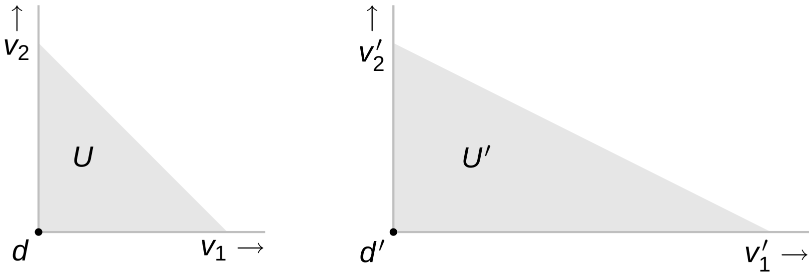
and let $d' = (\alpha_1 d_1 + \beta_1, \alpha_2 d_2 + \beta_2)$. Then

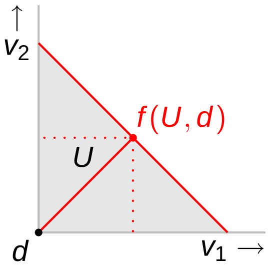
$$f_i(U', d') = \alpha_i f_i(U, d) + \beta_i \text{ for } i = 1, 2.$$

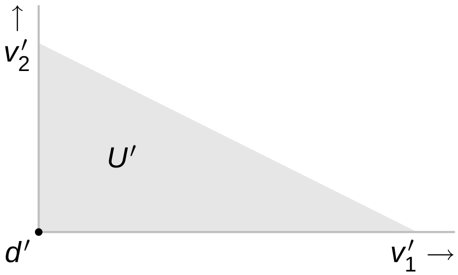
Nash's axiomatic model of bargaining: Axioms



Nash's axiomatic model of bargaining: Axioms

PAR & SYM $\Rightarrow$





Nash's axiomatic model of bargaining: Axioms

$$U' = \{(2v_1, v_2) : (v_1, v_2) \in U\}$$
and $d' = (2d_1, d_2)$,

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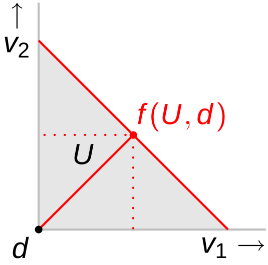

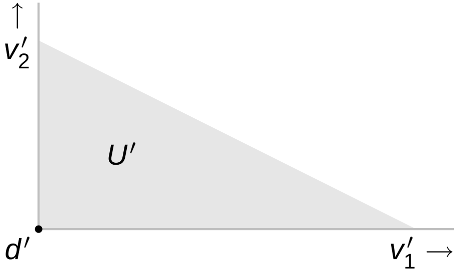

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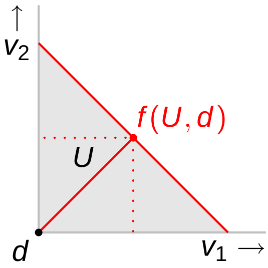
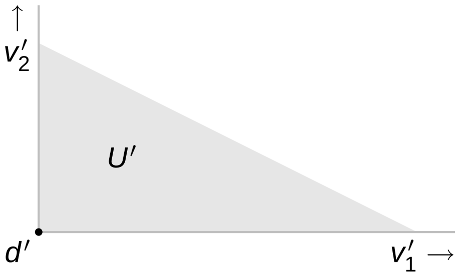


so PAR & SYM & INV $\Rightarrow$



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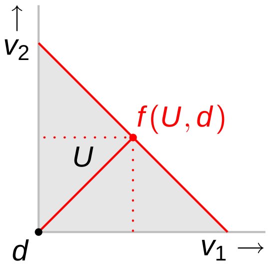

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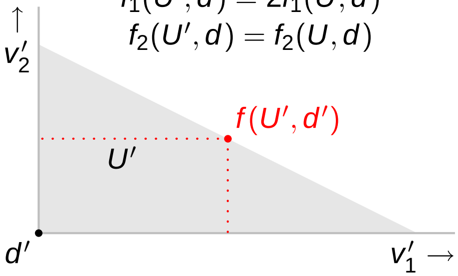

Nash's axiomatic model of bargaining: Axioms

- INV extends implications of PAR and SYM to affine transformations of symmetric problems

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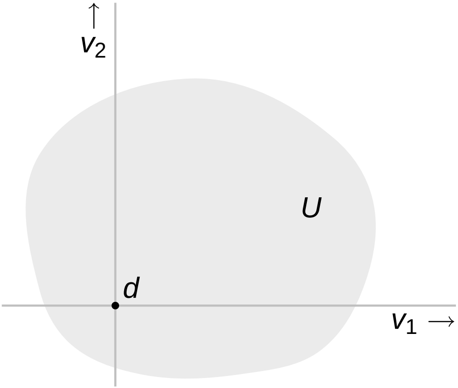
Independence of irrelevant alternatives (IIA)



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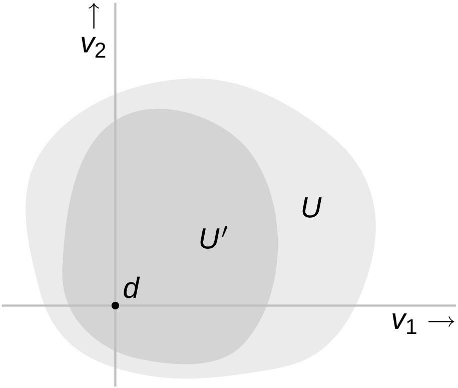

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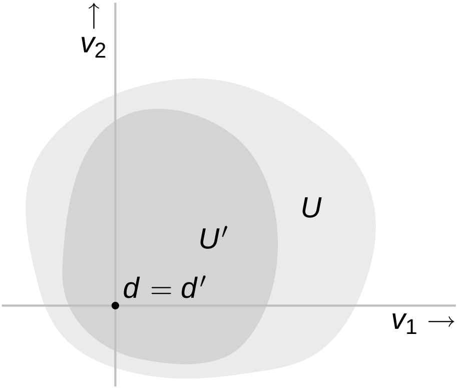
If $U' \subseteq U$, $d' = d$



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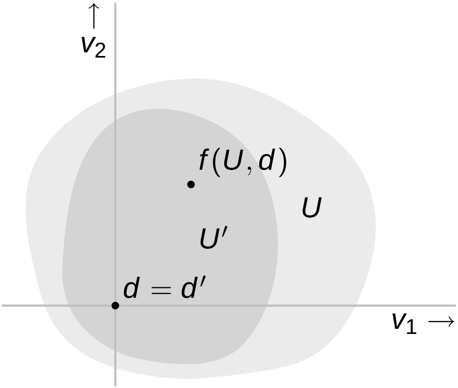
If $U' \subseteq U$, $d' = d$, and $U' \ni f(U, d)$



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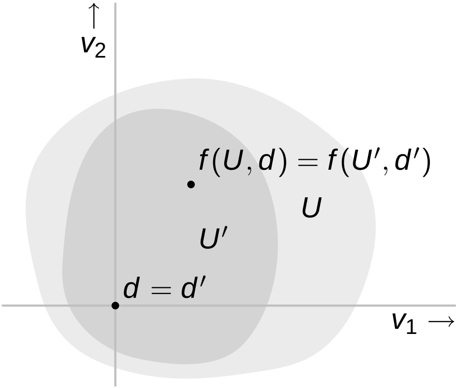

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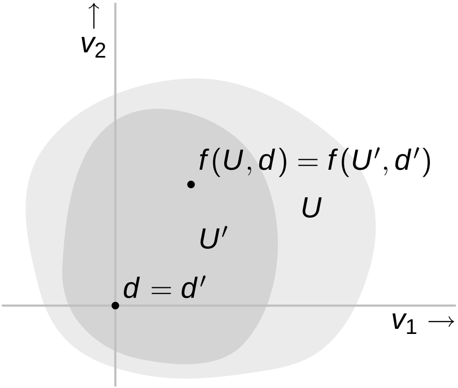

Idea: if $f(U,d) \in U'$ then members of $U \setminus U'$ are irrelevant

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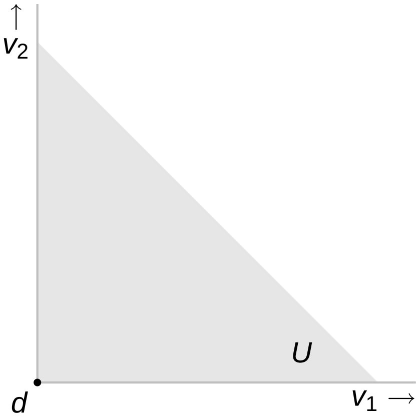
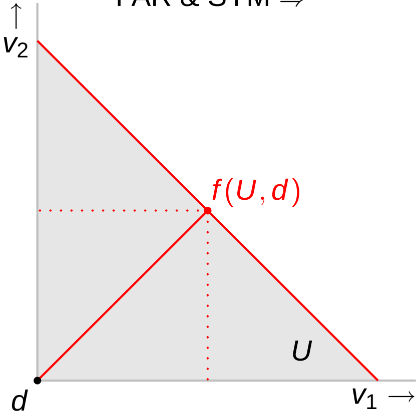
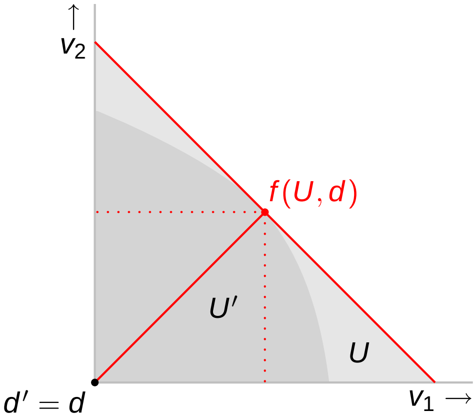


Nash's axiomatic model of bargaining: Axioms

PAR & SYM ⇒

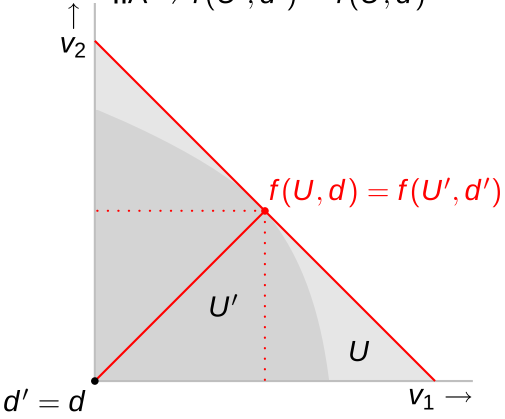


Nash's axiomatic model of bargaining: Axioms



Nash's axiomatic model of bargaining: Axioms

$$\text{IIA} \Rightarrow f(U', d') = f(U, d)$$



Nash Bargaining Solution

Proposition

A unique bargaining solution satisfies the axioms INV, SYM, IIA, and PAR.

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$$\mathcal{N}(U,d) = \arg\max_{(v_1,v_2)} (v_1 - d_1)(v_2 - d_2)$$
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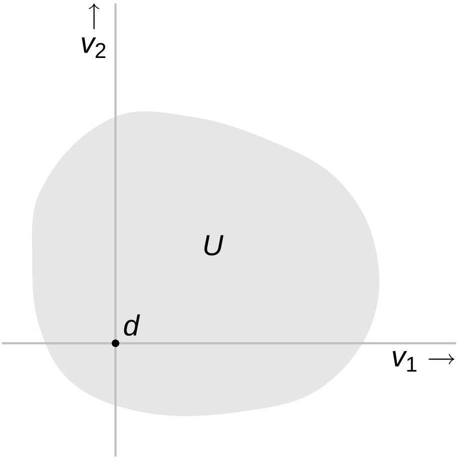
$\mathcal{N}(U,d)$ is the Nash solution of the bargaining problem (U,d)

Nash Bargaining Solution



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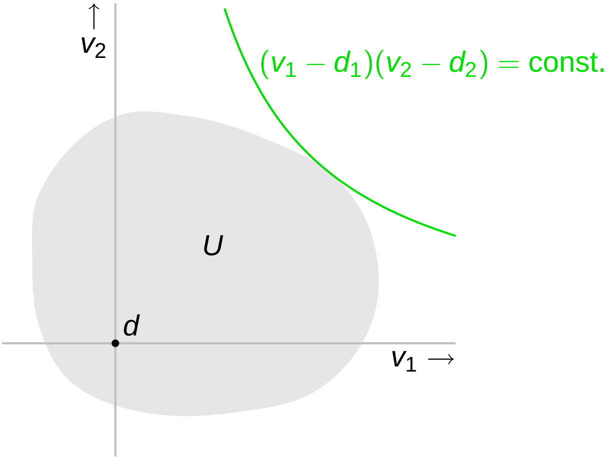

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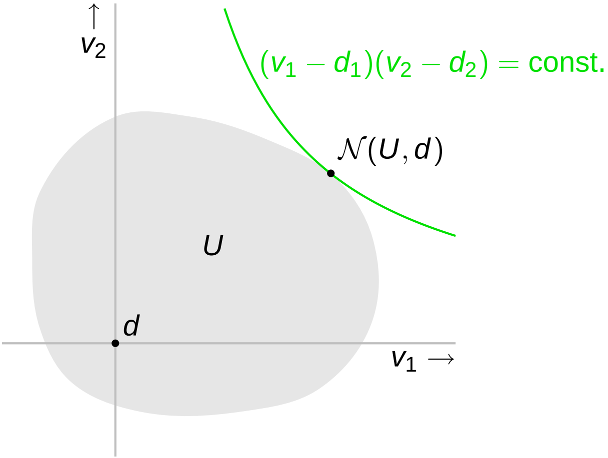

Nash Bargaining Solution: Proof of Proposition

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- Denote $z = \mathcal{N}(U, d)$ and let

$$\alpha_i = \frac{1}{2(z_i - d_i)} \quad \text{and} \quad \beta_i = \frac{-d_i}{2(z_i - d_i)} \quad \text{for } i = 1, 2$$

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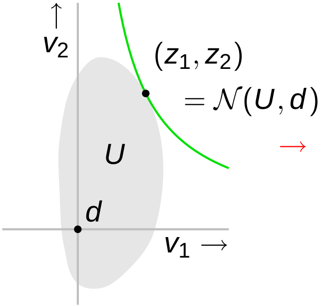
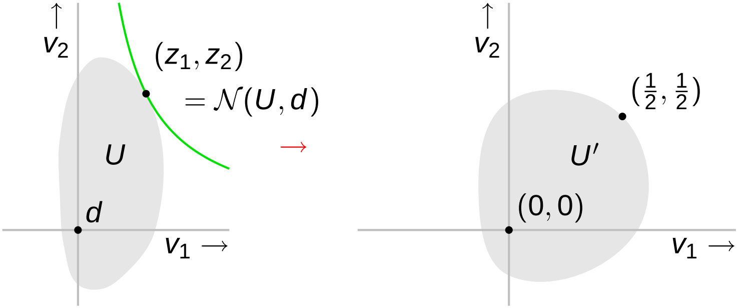


$\rightarrow$

Nash Bargaining Solution: Proof of Proposition



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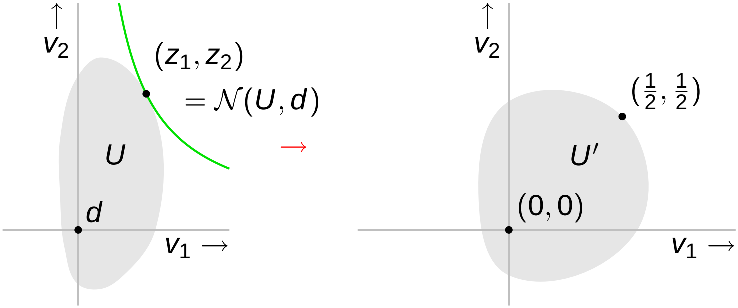

Outline of argument

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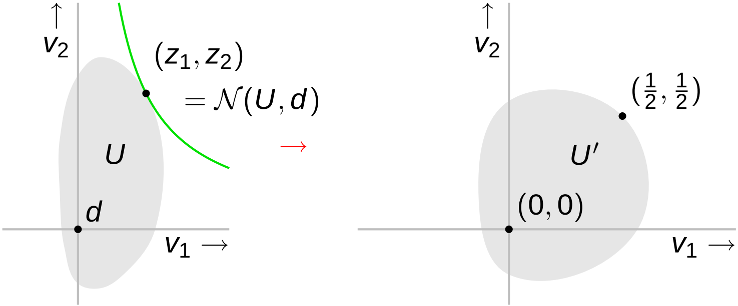

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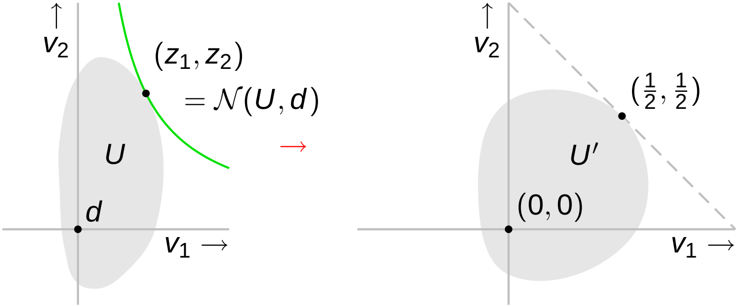

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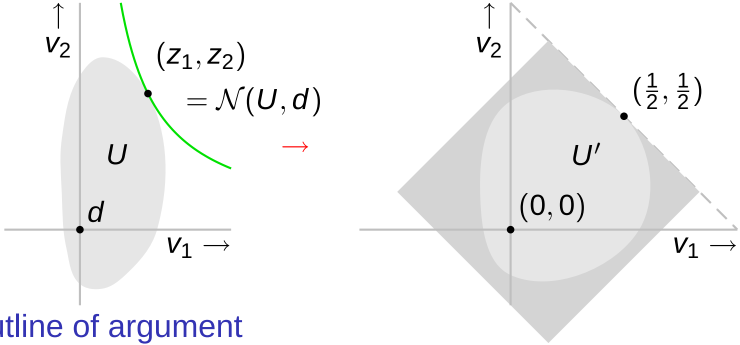

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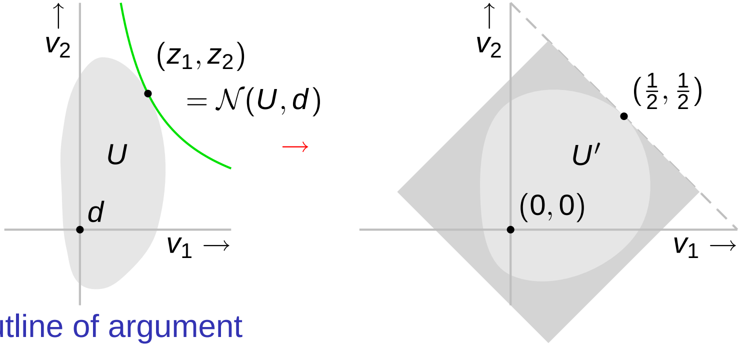

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$$\begin{aligned} & f_i(U', d') = \mathcal{N}_i(U', d') \text{ for } i = 1, 2 \\ & \qquad \Leftrightarrow \quad f_i(U, d) = \mathcal{N}_i(U, d) \text{ for } i = 1, 2. \end{aligned}$$

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$$\Leftrightarrow \quad f_i(U, d) = \mathcal{N}_i(U, d) \text{ for } i = 1, 2.$$

- That is, $f(U, d) = \mathcal{N}(U, d) \Leftrightarrow f(U', d') = \mathcal{N}(U', d')$

Nash Bargaining Solution: Proof of Proposition

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- So $\mathcal{N}(U', d') = (\frac{1}{2}, \frac{1}{2})$

Nash Bargaining Solution: Proof of Proposition

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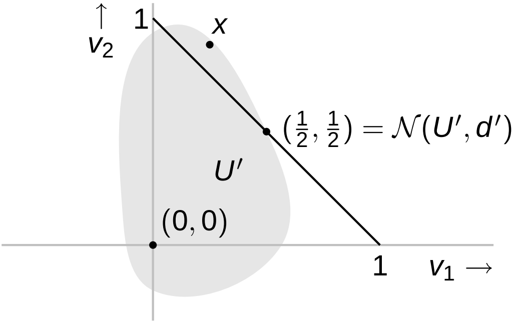

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- Suppose $(x_1, x_2) \in U'$ with $x_1 + x_2 > 1$
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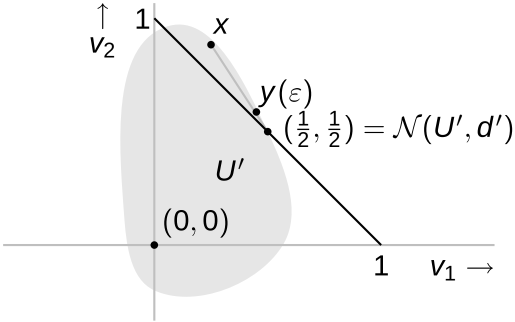

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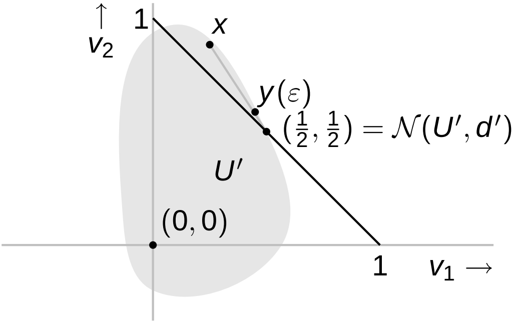

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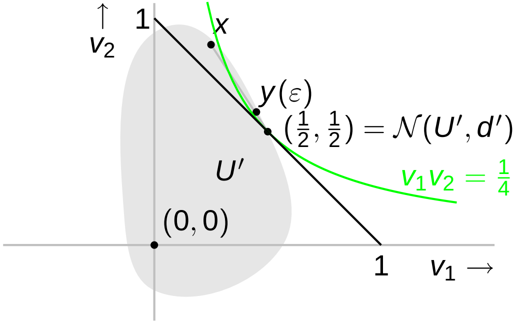
$\Rightarrow$ $y_1(\varepsilon)y_2(\varepsilon) > \frac{1}{4}$ for small ε, contradicting $\mathcal{N}(U', d') = (\frac{1}{2}, \frac{1}{2})$



Nash Bargaining Solution: Proof of Proposition

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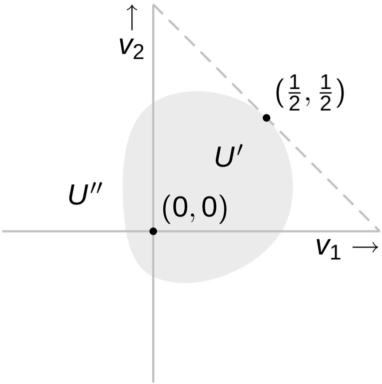
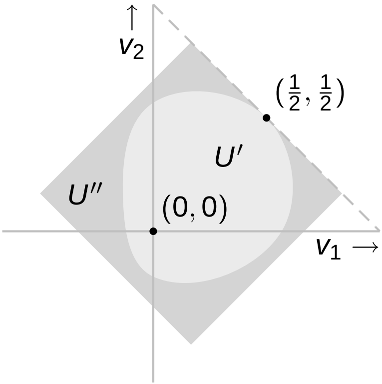


Nash Bargaining Solution: Proof of Proposition

Step 3 continued: U' is subset of symmetric set that includes $(\frac{1}{2}, \frac{1}{2})$

- Given U' contains no point (x_1, x_2) with $x_1 + x_2 = 1$, we can find symmetric rectangle U'' enclosing U' with Pareto surface intersecting $x_1 + x_2 = 1$



Nash Bargaining Solution: Proof of Proposition

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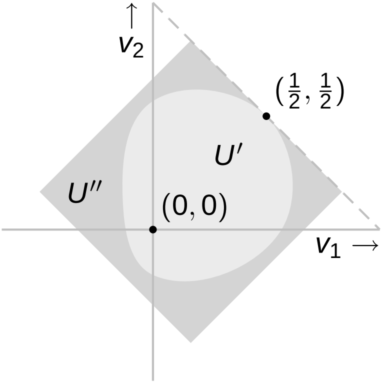

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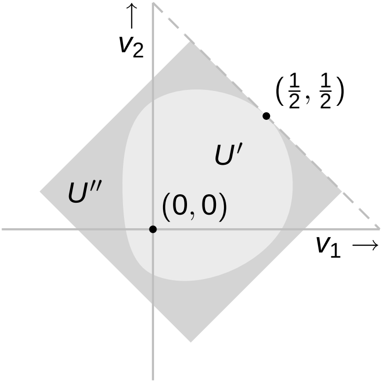
- By SYM and PAR we have $f(U'', d') = (\frac{1}{2}, \frac{1}{2})$
- By IIA we have $f(U', d') = (\frac{1}{2}, \frac{1}{2})$, completing the proof

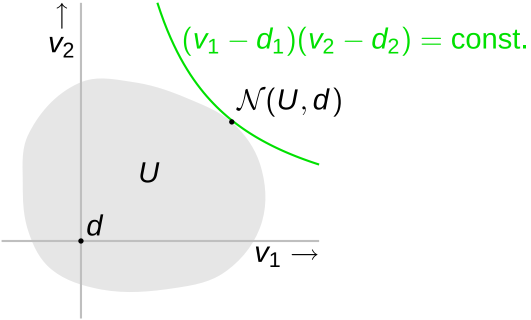


Nash Bargaining Solution

Proposition

A unique bargaining solution satisfies the axioms INV, SYM, IIA, and PAR. This solution is given by

$$\mathcal{N}(U,d) = \arg\max_{(v_1,v_2)} (v_1 - d_1)(v_2 - d_2)$$
$$\text{s.t. } (v_1,v_2) \in U \text{ and } (v_1,v_2) \geq (d_1,d_2).$$



Relation between strategic and axiomatic models

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- Thus (U, b) is a bargaining problem

Relation between strategic and axiomatic models

In bargaining game of alternating offers with risk of breakdown, SPE entails proposals $(\hat{x}_1(\alpha), \hat{x}_2(\alpha))$ and $(\hat{y}_1(\alpha), \hat{y}_2(\alpha))$ such that

$$u_1(\hat{y}_1(\alpha)) = (1-\alpha)u_1(\hat{x}_1(\alpha)) + \alpha b_1$$
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$$\Leftrightarrow$$

$(u_1(\hat{y}_1(\alpha)), u_2(\hat{y}_2(\alpha)))$ and $(u_1(\hat{x}_1(\alpha)), u_2(\hat{x}_2(\alpha)))$

lie on same rectangular hyperbola relative to axes through (b_1, b_2)

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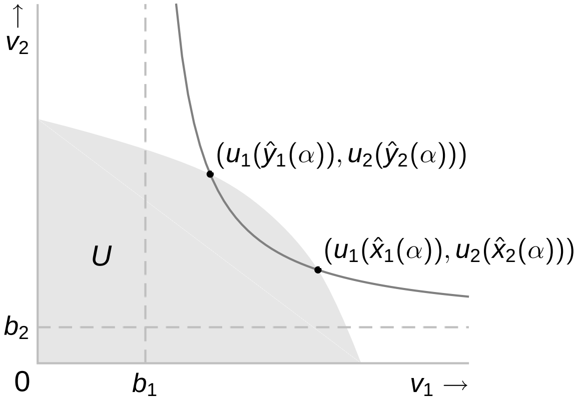

Relation between strategic and axiomatic models

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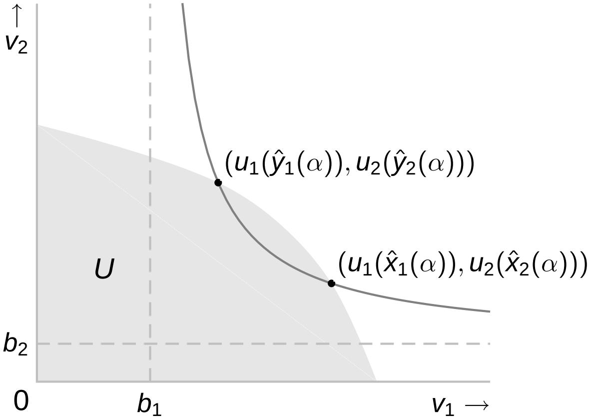

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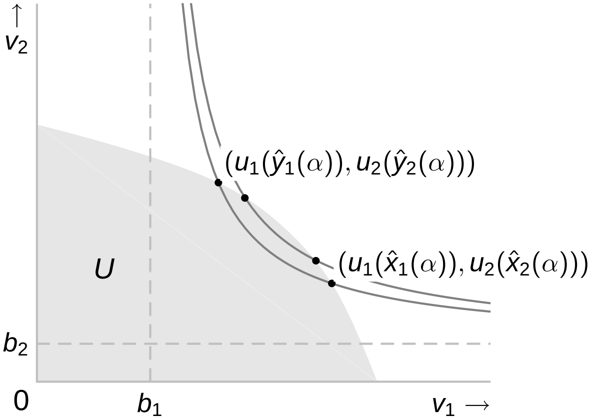

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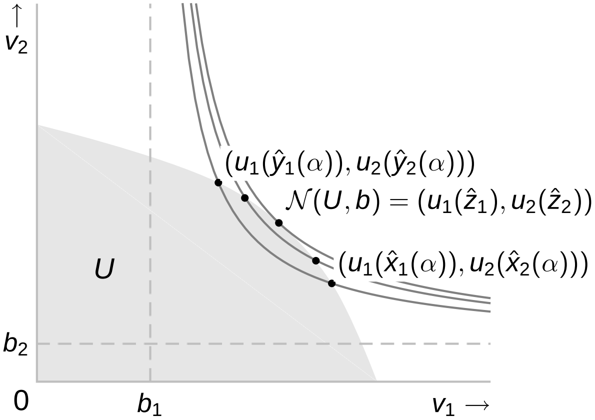

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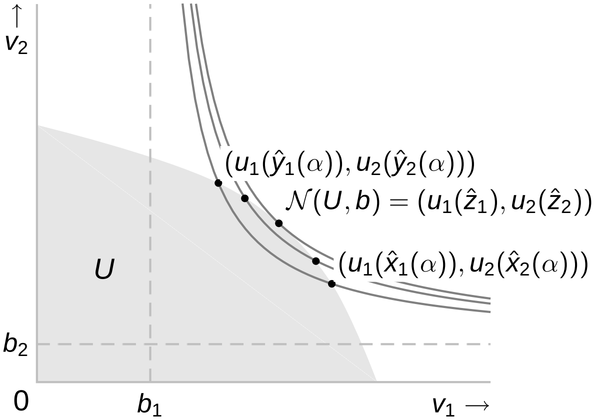

- Thus agreement $\hat{z}$ to which both $\hat{x}(\alpha)$ and $\hat{y}(\alpha)$ converge as $\alpha \to 0$ is Nash solution of (U, b)

Relation between strategic and axiomatic models

Proposition

The SPE outcome of the variant of the bargaining game of alternating offers with risk of breakdown (and discount factors of 1 for each player) converges to the Nash bargaining solution of the associated bargaining problem as the probability of breakdown converges to 0.

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- Bargaining game of alternating offers assumes specific bargaining procedure; Nash bargaining model does not
- Result sheds light on disagreement payoffs in Nash bargaining model: should be breakdown payoffs—and not, for example, payoffs players receive when they *choose* to leave the bargaining table

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- Disagreement point in Nash's model should be payoff in event of exogenous breakdown, *not* outside option payoff