Economics 2030

Fall 2018 Martin J. Osborne

Solutions to Problem Set 6

1. The game has six subgame perfect equilibria: (C, EG), (D, EG), (C, EH), (D, FG), (C, FH), (D, FH).

2. It is possible. The game in Figure 1 has a unique subgame perfect equilibrium (BE, C), with payoffs $(2, 2)$. The strategy pair (AF, D) is a Nash equilibrium, with payoffs $(3, 3)$.

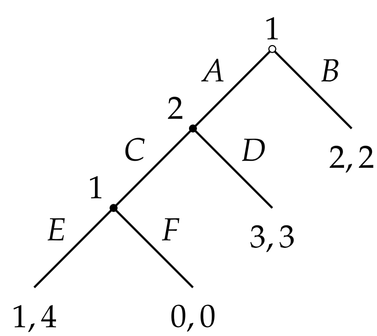

Figure 1. An extensive game with a Nash equilibrium in which each player is better off than she is in the unique subgame perfect equilibrium.

3. (a) Firm 2 chooses q_2 to solve

$$\max_{q_2} \left(\max\{0, (\alpha - q_1 - q_2)\} q_2 - q_2^2 \right),$$

so that $q_2 = \max\{0, (\alpha - q_1)/4\}$.
Firm 1 consequently chooses q_1 to solve

$$\max_{q_1} (\alpha - q_1 - \max\{0, (\alpha - q_1)/4\}) q_1 - q_1,$$

so that $q_1 = \frac{1}{2}\alpha - \frac{2}{3}$.
The equilibrium strategies are: $q_1 = \frac{1}{2}\alpha - \frac{2}{3}$ for firm 1, and $q_2 = (\alpha - q_1)/4$ for firm 2.
The equilibrium outcome is that $q_1 = \frac{1}{2}\alpha - \frac{2}{3}$ and $q_2 = \frac{1}{8}\alpha + \frac{1}{6}$.

(b) Such Nash equilibria exist.

Suppose, for example, that firm 2's strategy is to produce α units if $q_1 > 0$, and $\alpha/4$ if $q_1 = 0$. Then the price is 0 if firm 1 produces any positive output, so that its optimal output is 0. Further, given $q_1 = 0$, $q_2 = \alpha/4$ is optimal for firm 2. Thus the pair of strategies is a Nash equilibrium. The outcome is that firm 1's output is 0, while firm 2's is $\alpha/4$.

Another Nash equilibrium is (q_1^c, q_2^c), where q_1^c is firm 1's output in a Nash equilibrium of the simultaneous move game and q_2^c is the *function* that assigns to every output of firm 1 the output of firm 2 in a Nash equilibrium of the simultaneous move game.

4. (a) The following extensive game models the situation.

Players The firm and the union.

Histories $\varnothing$ and all sequences of the form w, (w, Y), (w, Y, L) and (w, N) for nonnegative numbers w and L (where w is a wage, Y means accept, N means reject, and L is the number of workers hired).

Player function $P(\varnothing)$ is the union, and, for any nonnegative number w, $P(w)$ and $P(w, Y)$ are the firm.

Preferences The firm's preferences are represented by its profit, and the union's preferences are represented by the value of wL (which is zero after any history (w, N)).

(b) First consider the subgame following a history (w, Y), in which the firm accepts the wage demand w. In a subgame perfect equilibrium, the firm chooses L to maximize its profit, given w. For $L \leq 50$ this profit is $L(100 - L) - wL$, or $L(100 - w - L)$. This function is a quadratic in L that is zero when $L = 0$ and when $L = 100 - w$ and reaches a maximum in between. Thus the value of L that maximizes the firm's profit is

$$\begin{cases} \frac{1}{2}(100 - w) & \text{if } w \leq 100 \\ 0 & \text{if } w > 100. \end{cases}$$

Given the firm's optimal action in such a subgame, consider the subgame following a history w, in which the firm has to decide whether to accept or reject w. For any w the firm's profit, given its subsequent optimal choice of L, is nonnegative; if $w < 100$ this profit is positive, while if $w \geq 100$ it is 0. Thus in a subgame perfect equilibrium, the firm accepts any demand $w < 100$ and either accepts or rejects any demand $w \geq 100$.

Finally consider the union's choice at the beginning of the game. If it chooses $w < 100$ then the firm accepts and chooses $L = (100 - w)/2$, yielding the union a payoff of $w(100 - w)/2$. If it chooses $w > 100$ then the firm either accepts and chooses $L = 0$ or rejects; in both cases the union's payoff is 0. Thus the best value of w for the union is the number that maximizes $w(100 - w)/2$. This function is a quadratic that is zero when $w = 0$ and when $w = 100$ and reaches a maximum in between; thus its maximizer is $w = 50$.

In summary, in a subgame perfect equilibrium the union's strategy is $w = 50$, and the firm's strategy accepts any demand $w < 100$ and chooses $L = (100 - w)/2$, and either rejects a demand $w \geq 100$ or accepts such a demand and chooses $L = 0$. The outcome of any equilibrium is that the union demands $w = 50$ and the firm chooses $L = 25$.

(c) Yes. In any subgame perfect equilibrium the union's payoff is $(50)(25) = 1250$ and the firm's payoff is $(25)(75) - (50)(25) = 625$. Thus both parties are better off at the outcome (w, L) than they are in the unique subgame perfect equilibrium if and only if $L \leq 50$ and

$$\begin{aligned} wL &> 1250 \\ L(100 - L) - wL &> 625 \end{aligned}$$

or $L \geq 50$ and

$$\begin{aligned} wL &> 1250 \\ 2500 - wL &> 625. \end{aligned}$$

These conditions are satisfied for a nonempty set of pairs (w, L). For example, if $L = 50$ the conditions are satisfied by $25 < w < 37.5$; if $L = 100$ they are satisfied by $12.5 < w < 18.75$.

(d) There are many Nash equilibria in which the firm "threatens" to reject high wage demands. In one such Nash equilibrium the firm threatens to reject any positive wage demand. In this equilibrium the union's strategy is $w = 0$, and the firm's strategy rejects any demand $w > 0$, and accepts the demand $w = 0$ and chooses $L = 50$. (The union's payoff is 0 no matter what demand it makes; given $w = 0$, the firm's optimal action is $L = 50$.)

5. (a) Fix an integer z with $0 \leq z \leq v_2$. For any history h, let $x(h)$ be the last bid. The game has a subgame perfect equilibrium (s_1, s_2) in which

$$s_1(h) = \begin{cases} z & \text{if } h = \varnothing \text{ or } x(h) < z \\ x(h) + 1 & \text{if } z \leq x(h) < v_1 \\ \text{quit} & \text{if } x(h) \geq v_1 \end{cases}$$

and

$$s_2(h) = \begin{cases} z & \text{if } x(h) < z \\ \text{quit} & \text{if } x(h) \geq z. \end{cases}$$

The outcome of such a subgame perfect equilibrium is that player 1 wins and pays the price z.

To show that this strategy pair is a subgame perfect equilibrium, I argue that it satisfies the one deviation property. First consider the actions of player 1 after a history h, given the strategy of player 2 and the rest of player 1's strategy.

- If $x(h) < z$ then she gets
  - 0 if she quits
  - $v_1 - (z + 1)$ if she bids between $x(h)$ and z
  - $v_1 - z$ if she bids z
  - $< v_1 - z$ if she bids more than z.

  Thus bidding z is optimal.
- If $z \leq x(h) < v_1$ then she gets
  - 0 if she quits
  - $v_1 - (x(h) + 1)$ if she bids $x(h) + 1$
  - $< v_1 - (x(h) + 1)$ if she bids more than $x(h) + 1$.

  Thus bidding $x(h) + 1$ is optimal.
- If $x \geq v_1$ then she gets 0 if she quits and a negative payoff if she bids.

  Thus quitting is optimal.

Now consider the actions of player 2 after a history h, given the strategy of player 1 and the rest of player 2's strategy.

- If $x(h) < z$ then she gets 0 whatever her action.
- If $x(h) \geq z$ then quitting yields 0 whereas any bid yields a payoff of at most 0.

Another subgame perfect equilibrium is the strategy pair in which after any history each player bids 1 more than the highest bid so far. The outcome is that play continues indefinitely, and the players' payoffs are zero. If, after any history, a player changes her strategy, the outcome either remains the same, or the player quits and obtains a payoff of zero.

(b) The game has such a Nash equilibrium. For example, the strategy pair defined as follows is an equilibrium:

$$s_1(h) = \text{quit} \quad \text{for all } h$$

and

$$s_2(h) = \begin{cases} 0 & \text{if } h = \varnothing \\ v_1 + 1 & \text{if } x(h) \leq v_1 \\ \text{quit} & \text{if } x(h) > v_1. \end{cases}$$

Given player 1's strategy, player 2's strategy is optimal—she obtains the good at the price of 0. Given player 2's strategy, player 1's strategy is optimal because following player 2's bid of zero, if she deviates to a bid of v_1 or less then player 2 bids $v_1 + 1$, so that player 1 cannot obtain a positive payoff, and if she deviates to a bid of more than v_1 then player 2 quits and player 1's payoff is negative.

In this equilibrium, player 2 obtains the good and pays the price 0.