Economics 2030

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Solutions to Problem Set 5

1. For any player i, the game has a Nash equilibrium in which player i bids $\overline{v}$ (the highest possible valuation) regardless of her valuation and every other player bids $\underline{v}$ regardless of her valuation. The outcome is that player i wins and pays $\underline{v}$. Player i can do no better by bidding less; no other player can do better by bidding more, because unless she bids at least $\overline{v}$ she does not win, and if she makes such a bid her payoff is at best zero. (It is zero if her valuation is $\overline{v}$, negative otherwise.)

2. The following argument is a variant of the one given in class for the case in which each player is risk neutral.

   The expected payoff of a player with valuation v who bids b when every other player's strategy is given by the bidding function β is

   $$(v-b)^{1/m}\Pr\{\text{Highest bid is } b\} = (v-b)^{1/m}\Pr\{\text{All other bids } \leq b\}.$$

   Now, any given player bids at most b if her valuation is at most $\beta^{-1}(b)$ (the inverse of β evaluated at the point b), so the probability that her bid is at most b is $F(\beta^{-1}(b)) = \beta^{-1}(b)$ (given that F is uniform on $[0,1]$. Hence the probability that the bid of all $n-1$ other players is at most b is $(\beta^{-1}(b))^{n-1}$. Thus the expected payoff in (1) is

   $$(v-b)^{1/m}(\beta^{-1}(b))^{n-1}.$$

   The best response of type v of any player when every other player uses the bidding function β is the value of b that maximizes this expected payoff, and hence satisfies the condition that the derivative of the function with respect to b is zero:

   $$-(1/m)(v-b)^{(1/m)-1}(\beta^{-1}(b))^{n-1} + (v-b)^{1/m}(n-1)(\beta^{-1}(b))^{n-2}/\beta'(\beta^{-1}(b)) = 0. \quad (1)$$

   (Recall that the derivative of β^{-1} at the point b is $1/\beta'(\beta^{-1}(b))$.)

Now, for $(\beta, \ldots, \beta)$ to be a Nash equilibrium, for every value of v the bid $\beta(v)$ must be the best response of a player with valuation v when every type v' of every other player bids $\beta(v')$. That is, $b = \beta(v)$ must satisfy (1). If $b = \beta(v)$ then $\beta^{-1}(b) = v$, so for all v we need

$$-(1/m)(v - \beta(v))^{(1/m)-1}v^{n-1} + (v - \beta(v))^{1/m}(n-1)v^{n-2}/\beta'(v) = 0$$

or

$$-(1/m)\beta'(v)v + (n-1)(v - \beta(v)) = 0.$$

To solve this differential equation, write it as

$$\beta'(v)v + m(n-1)\beta(v) = m(n-1)v,$$

multiply both sides by the integrating factor $v^{m(n-1)-1}$, and then integrate both sides, to get

$$v^{m(n-1)}\beta(v) = \left(\frac{m(n-1)}{m(n-1)+1}\right)v^{m(n-1)+1},$$

so that

$$\beta(v) = \left(\frac{m(n-1)}{m(n-1)+1}\right)v.$$

This function is increasing, so we conclude that the game has a Nash equilibrium in which each type v_i of each player i bids $(m(n-1)/[m(n-1)+1])v_i$.

In this equilibrium, the price paid by a bidder with valuation v who wins is $(1 - 1/[m(n-1)+1])v$ (the amount she bids). The expected price paid by a bidder in a second-price auction does not depend on the players' payoff functions. Thus this payoff is equal, by the revenue equivalence result, to the expected price paid by a bidder with valuation v who wins in a first-price auction in which each bidder is risk-neutral, namely $(1 - 1/n)v$. We have

$$\left(1 - \frac{1}{m(n-1)+1}\right) - \left(1 - \frac{1}{n}\right) = \frac{(m-1)(n-1)}{n(m(n-1)+1)},$$

which is positive because $m > 1$. Thus the expected price paid by a bidder with valuation v who wins is greater in a first-price auction than it is in a second-price auction. The probability that a bidder with any given valuation wins is the same in both auctions, so the auctioneer's expected revenue is greater in a first-price auction than it is in a second-price auction.

1

2

	0	1
0	$\frac{1}{2}, 0$	$0, -1$
1	$0, 0$	$0, -\frac{1}{2}$

2's valuation is 0 (p)

2

	0	1
0	$\frac{1}{2}, \frac{1}{2}$	$0, 0$
1	$0, 0$	$0, 0$

2's valuation is 1 ($1 - p$)

Figure 1. The first-price auction in Exercise 3.

3. The auctions may be modeled as Bayesian games as follows.

   Players The two bidders, say 1 and 2.

   States The two valuations 0 and 1 (of player 2).

   Actions Each player's set of actions is the set of possible bids (nonnegative numbers).

   Signals The signal function τ_1 of player 1 satisfies $\tau_1(0) = \tau_1(1)$ and the signal function τ_2 of player 2 satisfies $\tau_2(0) \neq \tau_2(1)$.

   Beliefs Player 1's belief is that the state is 0 with probability p and 1 with probability $1 - p$. Player 2's belief when her signal is $\tau_2(0)$ is that the state is 0 with probability 1, and her belief when her signal is $\tau_2(1)$ is that it is 1 with probability 1.

   Payoff functions Player 1's payoff is 0 if her bid is less than player 2's bid, $1 - P(b)$ if her bid is higher than player 2's bid, and $\frac{1}{2}(1 - P(b))$ if her bid is the same as player 2's bid, where $P(b)$ is either her bid (first-price auction) or player 2's bid (second-price auction). Player 2's payoff in state v is 0 if her bid is less than player 1's bid, $v - P(b)$ if her bid is higher than player 1's bid, and $\frac{1}{2}(v - P(b))$ if her bid is the same as player 1's bid.

   When each player is restricted to bid 0 or 1, the game that models a first-price auction is shown in Figure 1. The bid of 1 by player 2 of type 0 is strictly dominated by the bid of 0, so in any Nash equilibrium she bids 0. Thus in an equilibrium player 1 bids 0, and hence player 2 of type 1 bids 0. The auctioneer's revenue in this equilibrium is 0.

   In a second-price auction, each player's only strategy that is not weakly dominated is that in which she bids her valuation. The resulting strategy pair is a Nash equilibrium. In this equilibrium the auctioneer's revenue is $1 - p$ (the probability that player 2's valuation is 1, in which case the price is 1).

We conclude that the auctioneer's revenue is higher in the second-price auction than it is in the first-price auction.

4. (a) If player 2 wins, she knows that player 1 has bid at most 1, implying that the painting is fake.

   The strategy pair is not an equilibrium because any type $x_2 < 1$ of player 2 can profitably deviate, by the following argument. If type x_2 of player 2 bids $x_2 + 5$, she wins only if and only if the painting is fake; when she wins, she pays x_1, so that for $x_1 > x_2$ she pays more than her valuation. If she bids x_2, she wins only if the painting is fake *and* player 1's valuation is less than x_2; when she wins, she pays x_1, as when she bids $x_2 + 5$, but now whenever she wins her payoff is positive. Thus (a) the set of cases in which she wins if she bids x_2 is a subset of the set of cases in which she wins if she bids $b_2 + 5$, (b) in each case in which she wins if she bids x_2 she pays the same price as she does if she bids $b_2 + 5$, and (c) in every case in which she does not win if she bids x_2 (i.e. if $x_1 > x_2$) she pays more than her valuation if she bids $b_2 + 5$. Thus the expected payoff of any type $x_2 < 1$ of player 2 is higher if she bids x_2 than if she bids $x_2 + 5$.

   (b) Suppose that player 1 bids her valuation. Consider the optimal bid of type x_2 of player 2. Suppose that she bids b_2. Then if $b_2 \in [0, 1]$, she wins only if the painting is both fake, which occurs with probability $\frac{1}{2}$, and player 1 bids less than b_2, which occurs when player 1's valuation x_1 is less than b_2, and hence when the painting is fake occurs with probability b_2. If she wins in this case, her payoff is $x_2 - x_1$. The expected value of x_1 conditional on its being less than b_2 is $\frac{1}{2}b_2$, so if $b_2 \in [0, 1]$ then the expected payoff of type x_2 of player 2 is

$$\tfrac{1}{2}b_2(x_2 - \tfrac{1}{2}b_2).$$

   If $b_2 \in [10, 11]$ then type x_2 of player 2 wins if the painting is fake, regardless of player 1's valuation x_1, *or* if the painting is authentic and $x_1 < b_2$. Conditional on the painting's being fake, the expected value of player 1's bid is $\frac{1}{2}$. If the painting is authentic, the probability that $x_1 < b_2$ is $b_2 - 10$, and the expected value of player 1's valuation, and hence her bid, in this case is $\frac{1}{2}(10 + b_2)$. Thus the expected payoff of type x_2 of player 2 for $b_2 \in [10, 11]$ is

$$\tfrac{1}{2}(x_2 - \tfrac{1}{2}) + \tfrac{1}{2}(b_2 - 10)(10 + x_2 - \tfrac{1}{2}(10 + b_2)).$$

Thus the optimal bid of type x_2 of player 2, given player 1's strategy, is the value of b_2 that maximizes

$$\begin{cases} \frac{1}{2}b_2(x_2 - \frac{1}{2}b_2) & \text{if } b_2 \in [0, 1] \\ \frac{1}{2}(x_2 - \frac{1}{2}) + \frac{1}{2}(b_2 - 10)(10 + x_2 - \frac{1}{2}(10 + b_2)) & \text{if } b_2 \in [10, 11]. \end{cases}$$

The maximizer is x_2 if $x_2 \leq \frac{1}{2}$ and $x_2 + 10$ if $x_2 > \frac{1}{2}$.

For player 1, bidding her valuation is her only weakly undominated action, as in a standard independent private values second-price auction.

We conclude that the auction has an equilibrium in which each type x_1 of player 1 bids x_1 and type x_2 of player 2 bids x_2 if $x_2 \leq \frac{1}{2}$ and $x_2 + 10$ if $x_2 > \frac{1}{2}$.

5. (a) The game is specified as follows.

Players $\{1, 2\}$.

States The set of pairs (m_1, m_2) of amount of money, where $m_i \in [0, \infty)$ for $i = 1, 2$.

Actions The set of actions of each player i is the set $[0, \infty)$ of possible bids.

Signals The set of signals of each player i is $T_i = [0, \infty)$, and the signal function of each player i is $\tau_i(m_1, m_2) = m_i$.

Beliefs The prior belief of player i is any distribution that yields F_j as a posterior over m_j for every value of m_i.

Preferences The preferences of each player i are represented by the expected value of the Bernoulli payoff function that assigns the payoff $m_1 + m_2 - b_j$ if $b_i > b_j$, $\frac{1}{2}(m_1 + m_2 - b_j)$ if $b_i = b_j$, and 0 if $b_i < b_j$.

(b) Suppose that each type m_2 of player 2 bids km_2. Then if some type m_1 of player 1 bids b_1, her payoff when her opponent has type m_2 is

$$\begin{cases} m_1 + m_2 - km_2 & \text{if } b_1 > km_2 \\ 0 & \text{if } b_1 < km_2, \end{cases}$$

or, equivalently,

$$\begin{cases} m_1 + (1-k)m_2 & \text{if } m_2 < b_1/k \\ 0 & \text{if } m_2 > b_1/k. \end{cases}$$

She faces a distribution of possible opponents, so her expected payoff is the integral of this payoff over all possible values of m_2. If you draw a graph of the function, you can see that if $k < 1$ then the integral is increasing in m_2, so that it has no maximizer, whereas if $k > 1$ the value that maximized it is $m_1/(k-1)$. Alternatively, you can write the expected payoff of type m_1 of player 1 as

$$\int_0^{b_1/k} (m_1 + (1-k)m_2) f_2(m_2)\, dm_2.$$

This payoff is increasing in b_1 as long as the integrand is positive. Thus it is maximized for $b_1/k = m_1/(k-1)$.

That is, the best response of type m_1 of player 1 to a strategy profile in which each type m_2 of player 2 bids km_2 is the bid of $km_1/(k-1)$. Thus for the best response of player 1 to player 2's strategy to take the form $b_1 = km_1$ we need $k/(k-1) = k$, or $k = 2$, and the pair of strategies in which each type of each player i bids $2m_i$ is a Nash equilibrium.

(c) Suppose that player 2 of type m_2 bids $k_2 m_2$. Then by an argument like that in part (b), the optimal bid of type m_1 of player 1 is $k_2 m_1/(k_2 - 1)$. If this bid is to take the form $k_1 m_1$ then we need $k_1 = k_2/(k_2 - 1)$. Similarly, if player 1 of type m_1 bids $k_1 m_1$ then the optimal bid of type m_2 of player 2 is $k_1 m_2/(k_1 - 1)$, which takes the form $k_2 m_2$ if and only if $k_2 = k_1/(k_1 - 1)$. This condition is the same as $k_1 = k_2/(k_2 - 1)$ (or $k_1 k_2 = k_1 + k_2$), so any pair (k_1, k_2) that satisfies this equation yields an equilibrium. For example, $(3, \frac{3}{2})$ or $(4, \frac{4}{3})$.