Economics 2030

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Solutions for Questions in Tutorial 4

1. (a) The subgame following the history A is an ultimatum game. Its unique subgame perfect equilibrium is the strategy pair in which the official demands y and the agent agrees to pay any bribe of at most y. The equilibrium payoffs in this subgame are thus $-c$ to the agent and y to the official. Thus the whole game has a unique subgame perfect equilibrium, in which the agent chooses B at the start of the game, and the equilibrium payoffs of both the agent and the official are zero.

   (b) The game is shown in Figure 1. For any value of b_1, the subgame following the history (A, b_1, Y, A) is an ultimatum game. Its unique subgame perfect equilibrium is the strategy pair in which the official demands $(1-\alpha)y$ and the agent agrees to pay any bribe of at most $(1-\alpha)y$. The outcome in the subgame yields the agent the payoff $\alpha y - b_1 - c$ and the official the payoff $b_1 + (1-\alpha)y$.

   In the subgame following any history (A, b_1, Y), the agent is thus indifferent between A and B, in both cases obtaining the payoff $\alpha y - b_1 - c$. Her action at the start of this subgame affects the official's incentives at her first move. Suppose that for some number b_1^* with $(2\alpha - 1)y \leq b_1^* \leq \alpha y - c$ the agent chooses A after any history (A, b_1, Y) with $b_1 \leq b_1^*$ and B after any history (A, b_1, Y) with $b_1 > b_1^*$. (Such a number b_1^* exists because $\alpha \leq 1 - c/y$.)

   Now, following any history (A, b_1), the agent obtains $\alpha y - b_1 - c$ if she chooses Y and $-c$ if she chooses N. Thus she chooses Y if $b_1 < \alpha y$ and N if $b_1 > \alpha y$. If $b_1 = \alpha y$, she is indifferent between Y and N. In the subgame perfect equilibrium I construct, she chooses Y.

   Finally, suppose that the official chooses the bribe b_1^* after the history A, and the agent chooses A at the start of the game.

   That is, consider the following strategy pair.

   Agent • A at start of game.

- Y after any history (A, b_1) with $b_1 \leq \alpha y$ and N after any history (A, b_1) with $b_1 > \alpha y$.
- A after any history (A, b_1, Y) with $b_1 \leq b_1^*$ and B after any history (A, b_1, Y) with $b_1 > b_1^*$.
- Y after any history (A, b_1, Y, A, b_2) with $b_2 \leq (1-\alpha)y$ and N after any history (A, b_1, Y, A, b_2) with $b_2 > (1-\alpha)y$.

Official
- b_1^* after the history A.
- $(1-\alpha)y$ after any history (A, b_1, Y, A).

I claim that this strategy pair is a subgame perfect equilibrium. I argue that it satisfies the one deviation property. It generates the outcome $(A, b_1^*, Y, A, (1-\alpha)y, Y)$, yielding the agent the payoff $\alpha y - b_1^* - c$ and the official the payoff $b_1^* + (1-\alpha)y$.

We have $b_1^* \leq \alpha y - c$, so the agent cannot increase her payoff by switching from A to B at the start of the game.

If the official reduces the value of b_1, her payoff decreases to $b_1 + (1-\alpha)y$. If she increases the value of b_1, her payoff changes to b_1 if $b_1 \leq \alpha y$ (because the agent responds with Y, then A) and changes to 0 if $b_1 > \alpha y$. Thus the deviation that yields the highest payoff is $b_1 = \alpha y$, which yields the payoff αy. The official's payoff in the strategy pair is $b_1^* + (1-\alpha)y$, so the deviation is not profitable if $b_1^* + (1-\alpha)y \geq \alpha y$, or $b_1^* \geq (2\alpha - 1)y$.

The agent's actions after a history (A, b_1) are optimal because Y generates the payoff $\alpha y - b_1^* - c$ for her and N generates the payoff $-c$.

The agent's actions after a history (A, b_1, Y) are also optimal, because both A and B generate the same payoff, namely $\alpha y - b_1^* - c$.

Finally, the subgame following any history (A, b_1, Y, A) is an ultimatum game, and hence the specified strategies form a subgame perfect equilibrium.

We conclude that the strategy pair is a subgame perfect equilibrium of the whole game.

By allowing the agent to quit midway through the activity, the official cedes her enough power to induce her to pursue A.

The game has also a subgame perfect equilibrium in which the agent chooses B at the start of the game. Consider, for example, the strategy pair that differs from the one just described only in that the agent chooses A after every history (A, b_1, Y), the official chooses $b_1 = \alpha y$ after the history A, and the agent chooses B

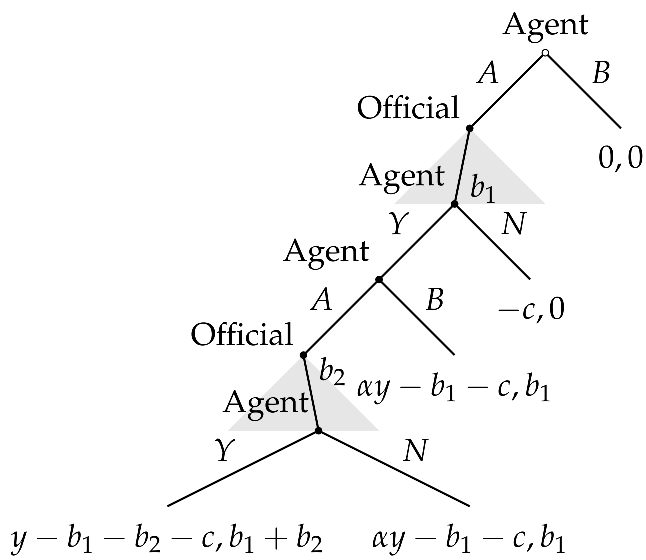

Figure 1. The game in part *b* of Exercise 1.

at the start of the game. This strategy pair is a subgame perfect equilibrium.

Source: Ng, Travis (2011), "Destructing the hold-up", *Economics Letters* **111**, 247–248.

2. (a) First consider period T (the last), after any history.

- If the incumbent chooses to fight, the challenger obtains $-F$ in the period if it stays and 0 if it exits, so exiting is optimal.
- If the incumbent chooses to cooperate, the challenger obtains C in the period if stays and 0 if it exits, so staying is optimal, given $C > 0$.
- Given the challenger's optimal actions, the incumbent obtains 0 in the period if it fights and C if it cooperates, so that cooperation is optimal.

Now consider period $T - 1$.

- If the incumbent fights, a challenger who stays in gets $-F$ in period $T - 1$ and C in period T, for a total of $C - F > 0$, and a challenger who exits gets 0. Thus after a fight by the incumbent the challenger optimally stays, and the incumbent's payoff in the last two periods is $C - F$.
- If the incumbent cooperates, a challenger who stays in gets C in both period $T - 1$ and period T, for a total of $2C$, and

a challenger who exits gets 0. Thus after the incumbent cooperates the challenger optimally stays, and the incumbent's payoff in the last two periods is $2C$.

- Given the challenger's optimal actions, the incumbent optimally cooperates in period $T-1$.

Now use induction. Suppose that from period $T-K$ on, the incumbent cooperates after any history and the challenger stays after any history up to period T, when it exits if the incumbent fights and stays if the incumbent cooperates. Now consider period $T-K-1$.

- If the incumbent fights, a challenger who stays in gets $-F$ in period $T-K-1$ and C in every subsequent period, for a total of $(K-1)C-F>0$, and a challenger who exits gets 0. Thus after a fight by the incumbent the challenger optimally stays, and the incumbent's payoff in the remaining periods is $(K-1)C-F$.
- If the incumbent cooperates, a challenger who stays in gets C in the remaining K periods, for a total of KC, and a challenger who exits gets 0. Thus after the incumbent cooperates the challenger optimally stays, and the incumbent's payoff in the remaining periods is KC.
- Given the challenger's optimal actions, the incumbent optimally cooperates in period $T-K-1$.

Thus the payoff to the challenger if it enters is $TC-f$. Given $C>f$, the challenger optimally enters at the start of the game.

That is, the game has a unique subgame perfect equilibrium, in which

- the challenger enters at the start of the game, exits in the last period if the challenger fights in that period, and stays in after every other history after which it moves
- the incumbent cooperates after every history after which it moves.

The incumbent's payoff in this equilibrium is TC and the challenger's payoff is $TC-f$.

(b) First consider the incumbent's action after the history in which the challenger enters, the incumbent fights in the first $T-2$ periods, and in each of these periods the challenger stays in. Denote this history h_{T-2}.

- If the incumbent fights after h_{T-2}, the challenger exits (it has no alternative), and the incumbent's total payoff in the last two periods is M.
- If the incumbent cooperates after h_{T-2}, then by the argument for the game in part a, the challenger stays in, and in the last period the incumbent also cooperates and the challenger stays in. Thus the incumbent's payoff in the last two periods is $2C$.

Because $M > 2C$, we conclude that the incumbent fights after the history h_{T-2}.

Now consider the incumbent's action after the history in which the challenger enters, the incumbent fights in the first $T-3$ periods, and in each period the challenger stays in. Denote this history h_{T-3}.

- If the incumbent fights after h_{T-3}, we know, by the previous paragraph, that if the challenger stays in then the incumbent will fight in the next period, driving the challenger out. Thus the challenger will obtain an additional profit of $-F$ if it stays in and 0 if it exits. Consequently the challenger exits if the incumbent fights after h_{T-3}, yielding the incumbent the payoff $2M$.
- Now suppose the incumbent cooperates after h_{T-3}. Then if the challenger stays in, it obtains a payoff of C in period $T-2$, and by the argument for part a, payoffs of C in each of the following periods. (Note that the histories after which it acts contain fewer than $T-1$ fights, so that it does not have to exit.) Thus a challenger who stays in obtains the payoff $3C$. A challenger who exits obtains the payoff 0. Thus the challenger optimally stays, yielding the incumbent the payoff $3C$ in the last three periods.

We conclude that after the history h_{T-3} the incumbent fights (given $M > 2C$).

Working back to the first period we conclude that the incumbent fights and the challenger exits. (Formally, you need to use induction.) Thus the challenger's optimal action at the start of the game is to stay out.

In summary, the game has a unique subgame perfect equilibrium, in which

- the challenger stays out at the start of the game, exits after any history in which the incumbent fought in every period, exits in the last period if the incumbent fights in that period, and stays in after every other history.
- the incumbent fights after the challenger enters and after any history in which it has fought in every period, and cooperates after every other history.

The incumbent's payoff in this equilibrium is TM and the challenger's payoff is 0.

Source: Benoît, Jean-Pierre (1984), "Financially constrained entry in a game of incomplete information", *Rand Journal of Economics* **15**, 490–499.

3. (a) The subgame following a proposal (x_1, x_2) with $x_1 > 0$ has two Nash equilibria, (N, N) and (Y, N), both resulting in the payoff pair $(0, 1)$.

   The subgame following the proposal $(0, 1)$ has four Nash equilibria, (Y, Y), (Y, N), (N, Y), and (N, N), all resulting in the payoff pair $(0, 1)$.

   Now consider the whole game. Let X be the set of possible proposals. A strategy profile is a pair $((x_1, x_2), V_1), V_2)$, where (x_1, x_2) is a proposal and $V_i : X \to \{Y, N\}$ is a voting function for $i = 1$, 2. Whatever player 1 proposes, the resulting payoff pair is $(0, 1)$. Thus a strategy pair $((x_1, x_2), V_1), V_2)$ is a subgame perfect equilibrium if and only if $V_2(x_1, x_2) = N$ whenever $x_1 > 0$.

   (b) For a strategy pair to generate payoffs different from $(0, 1)$, player 1 must make a proposal $(x, 1 - x)$ with $x > 0$ and both players must vote in favor of this proposal. But if player 2 deviates from this strategy and votes against the proposal her payoff increases from $1 - x$ to 1. Thus no Nash equilibrium exists in which the players' payoffs differ from $(0, 1)$.

   (c) From part (a), in every subgame perfect equilibrium of the subgame following player 1's initial offer being rejected, the payoff profile is $(0, 0, 1)$.

   Suppose that player 1's proposal is (x_1, x_2, x_3), with $x_1 > 0$ and $x_2 > 0$. The subgame following this proposal has two types of subgame perfect equilibria:

   - players 1 and 2 vote Y, and the payoff profile is (x_1, x_2, x_3)

- all three players vote N, and the payoff profile is $(0, 0, 1)$.

(In the first type of equilibrium, a change in the vote of player 1 or player 2 makes the deviator worse off and a change in the vote of player 3 has no effect on the outcome. In the second type of equilibrium, a change in the vote of any player has no effect on the outcome.)

If player 1's proposal (x_1, x_2, x_3) has $x_1 = 0$ or $x_2 = 0$, or both, then the subgame has more equilibria, though every equilibrium still yields either the payoff profile (x_1, x_2, x_3) or the payoff profile $(0, 0, 1)$.

Now consider the whole game. I claim that for *any* proposal x, the game has a subgame perfect equilibrium in which player 1 proposes x. One equilibrium strategy profile is

- player 1 proposes x
- for $i = 1, 2, 3$,

$$V_i(y) = \begin{cases} Y & \text{if } y = x \\ N & \text{if } y \neq x \end{cases}$$

- in every subgame following the rejection of player 1's proposal, players 2 and 3 follow their subgame perfect equilibrium strategies in the two-player version of the game.

To verify that this strategy profile is a subgame perfect equilibrium, check that it satisfies the one-deviation property, as follows.

- If player 1 proposes $y \neq x$ at the start of the game, this proposal is rejected, and she obtains the payoff 0 rather than $x_1 \geq 0$.
- In any subgame following a proposal of player 1, the strategy profile calls for all players to vote in the same way. If any player deviates, the outcome remains the same.
- In any subgame following the rejection of player 1's proposal, the remaining players follow the subgame perfect equilibrium strategies in a two-player game.

In this equilibrium, any proposal of player 1 in which she gets more than x_1 is rejected, and player 1 obtains the payoff 0. Given that both player 2 and player 3 reject such a proposal, it is optimal for player 1 to do so. Note, however, that any strategy of player 1 that rejects such a proposal is weakly dominated by a strategy that differs only in that she accepts such a proposal.

Thus any proposal of player 1 is consistent with a subgame perfect equilibrium of the three-player game.

4. Look for a stationary equilibrium, in which player 1 always make the same proposal, x, player M always makes the same proposal, y, player 1 accepts a proposal z if and only if it $z_1 \geq y$, and player M accepts a proposal z if and only if $z_M \geq x$. In such an equilibrium, player 1 must be indifferent between accepting and rejecting the proposal y and player M must be indifferent between accepting and rejecting the proposal x.

   If player 1 accepts y, then her payoff is y_1. If she rejects y (but otherwise behaves according to her strategy), then player M chooses to continue bargaining with her and she proposes x, which player M accepts, yielding player 1 the payoff δx_1. Thus for an equilibrium we need

$$y_1 = \delta x_1.$$

   If player M accepts x, then her payoff is $x_M + \delta(\delta/(1+\delta))$ (given that her payoff in $\Gamma(2, M)$ is $\delta/(1+\delta)$). If she rejects x, then she proposes y, which player 1 accepts, so that her payoff is $\delta y_M + \delta^2(\delta/(1+\delta))$. Thus for an equilibrium we need

$$x_M + \delta(\delta/(1+\delta)) = \delta y_M + \delta^2(\delta/(1+\delta)).$$

   Using the fact that $x_1 + x_M = 1$ and $y_1 + y_M = 1$, the second equation is

$$1 - x_1 + \delta^2/(1+\delta) = \delta(1 - \delta x_1) + \delta^3/(1+\delta),$$

   which leads to

$$x_1 = \frac{1+\delta+\delta^2}{(1+\delta)^2}, \qquad x_M = \frac{\delta}{(1+\delta)^2}$$

   and

$$y_1 = \frac{\delta(1+\delta+\delta^2)}{(1+\delta)^2}, \qquad y_M = \frac{1+\delta-\delta^3}{(1+\delta)^2}.$$

   The strategy pair satisfies the one-deviation property, and hence is a subgame perfect equilibrium.