ECO2030: Microeconomic Theory II, module 1

Lecture 8

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Bargaining

Finite horizon

Infinite horizon
- SPE
- Characterization of SPE
- Properties of SPE
- Many players

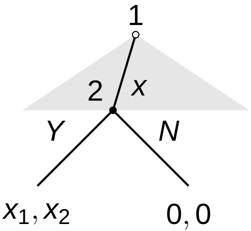
Risk of breakdown

Outside options

Outside options vs. breakdown

Bargaining

Ultimatum game with pie of size 1



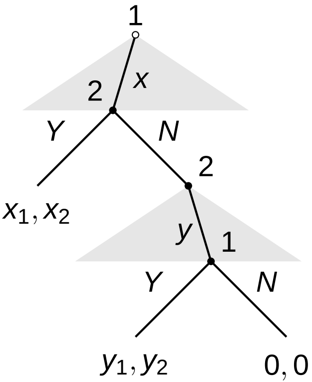
Unique SPE

Offer $(1, 0)$

Accept all offers

- SPE payoffs: $(1, 0)$
- Why is SPE outcome so one-sided?
- Should give player 2 the opportunity to counteroffer?

Bargaining: Two-period game



SPE

Any offer — Any offer

Reject all offers — Reject all offers except $(0, 1)$

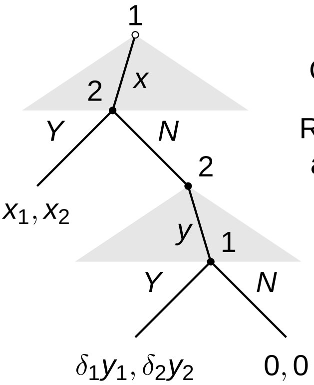
Offer $(0, 1)$

Accept all offers

$\Rightarrow$ in every subgame perfect equilibrium payoffs are $(0, 1)$

Bargaining: Two-period game with cost of delay

- Extreme outcome is result of delay being costless?
- Suppose delay is costly: players discount payoffs



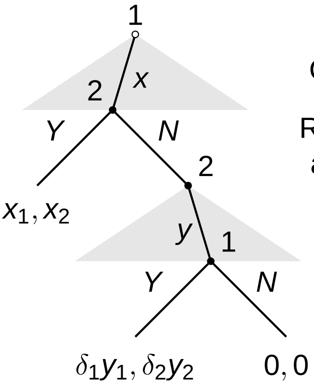
SPE

Offer $(1-\delta_2, \delta_2)$	No optimal offer
Reject x if $x_2 < \delta_2$ accept if $x_2 \geq \delta_2$	Reject x if $x_2 \leq \delta_2$ accept if $x_2 > \delta_2$

Offer $(0, 1)$

Accept all offers

Bargaining: Two-period game with cost of delay



SPE

Offer $(1-\delta_2, \delta_2)$	No optimal offer
Reject x if $x_2 < \delta_2$ accept if $x_2 \geq \delta_2$	Reject x if $x_2 \leq \delta_2$ accept if $x_2 > \delta_2$

Offer $(0, 1)$

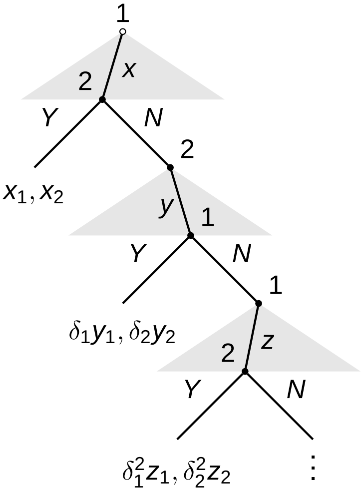
Accept all offers

Unique SPE

Period 1 P1 proposes $(1-\delta_2, \delta_2)$, P2 accepts $(x_1, x_2) \Leftrightarrow x_2 \geq \delta_2$

Period 2 P2 always proposes $(0, 1)$, P1 always accepts all proposals

Bargaining: Infinite horizon



Notes

- Every subgame starting with proposal by P1 is *identical*
- Every subgame starting with proposal by P2 is identical
- Every subgame starting with response by P1 to proposal *y* of P2 is identical
- Every subgame starting with response by P2 to proposal *x* of P1 is identical

Bargaining: Infinite horizon

A bargaining game of alternating offers is an extensive game with perfect information with the following components

Players $N = \{1, 2\}$

Histories $\varnothing$

every sequence $(x^1, N, x^2, N, \ldots, x^t)$, $t \geq 1$
every sequence $(x^1, N, x^2, N, \ldots, x^t, Y)$, $t \geq 1$
every sequence $(x^1, N, x^2, N, \ldots, x^t, N)$, $t \geq 1$
every infinite sequence $(x^1, N, x^2, N, \ldots)$
where each x^r is pair of numbers that sum to 1

Player function $P(\varnothing) = 1$ and for all $(x^1, \ldots, x^t)$

$$P(x^1, N, x^2, N, \ldots, x^t) = P(x^1, N, x^2, N, \ldots, x^t, N) = \begin{cases} 1 & \text{if } t \text{ is even} \\ 2 & \text{if } t \text{ is odd} \end{cases}$$

Preferences For $i = 1$, 2, player i's payoff to terminal history $(x^1, N, x^2, N, \ldots, x^t, Y)$ is $\delta_i^{t-1} x_i^t$ $(0 < \delta_i < 1)$, and her payoff to every (infinite) terminal history $(x^1, N, x^2, N, \ldots)$ is 0

Bargaining game of alternating offers: SPE

Step 1

- Stationary structure of game suggests SPE exists in which
  - P1 always makes same proposal
  - P2 always makes same proposal
  - P1 always responds to a proposal in the same way
  - P2 always responds to a proposal in the same way
- Reasonable guess: game has SPE in which each player's response has cutoff form: accept high offers, reject low ones
- Any strategy pair that satisfies these conditions has form
  - player 1 always proposes x^* and accepts y iff $y_1 \geq y_1^*$
  - player 2 always proposes z^* and accepts w iff $w_2 \geq w_2^*$

  for some proposals w^*, x^*, y^*, and z^*

Bargaining game of alternating offers: SPE

Step 2

- Can we find proposals w^*, x^*, y^*, and z^* such that following strategy pair is SPE?
  - Player 1 always proposes x^* and accepts y iff $y_1 \geq y_1^*$
  - Player 2 always proposes z^* and accepts w iff $w_2 \geq w_2^*$
- In SPE of two-period game, every equilibrium proposal is accepted . . . guess that infinite game has equilibrium with same property

  $\Rightarrow$ $x_2^* \geq w_2^*$ and $z_1^* \geq y_1^*$
- Is SPE with $x_2^* > w_2^*$ possible?
  - $x_2^* > w_2^* \Rightarrow$ P2 willing to accept less than she is offered

  $\Rightarrow$ P1 can increase payoff by reducing offer

  $\Rightarrow$ In SPE, $x_2^* = w_2^*$
  - Similarly $z_1^* = y_1^*$

Bargaining game of alternating offers: SPE

Step 3

- Can we find proposals x^* and y^* such that following strategy pair is SPE?
  - Player 1 always proposes x^* and accepts y iff $y_1 \geq y_1^*$
  - Player 2 always proposes y^* and accepts x iff $x_2 \geq x_2^*$
- Consider subgame in which first move is response by P2 to proposal x of P1
  - If P2 accepts x, her payoff is x_2
  - If P2 rejects x, she proposes y^*, which P1 accepts, yielding P2 the payoff y_2^* with one period of delay
  - So P2 optimally
    - rejects x if $x_2 < \delta_2 y_2^*$
    - accepts x if $x_2 > \delta_2 y_2^*$
    - is indifferent between accepting and rejecting if $x_2 = \delta_2 y_2^*$
- We seek SPE in which she accepts iff $x_2 \geq x_2^*$, so we need $x_2^* = \delta_2 y_2^*$

Bargaining game of alternating offers: SPE

Step 3 continued

- Similar argument for response by P1 to proposal of P2
  $\Rightarrow y_1^* = \delta_1 x_1^*$
- Thus for strategy pair to be SPE we need

$$x_2^* = \delta_2 y_2^*$$
$$y_1^* = \delta_1 x_1^*$$

- Using $x_2^* = 1 - x_1^*$ and $y_2^* = 1 - y_1^*$, we get

$$x_1^* = \frac{1 - \delta_2}{1 - \delta_1 \delta_2}$$
$$y_1^* = \frac{\delta_1 (1 - \delta_2)}{1 - \delta_1 \delta_2}$$

Bargaining game of alternating offers: SPE

Conclusion

Candidate for SPE is strategy pair s^* in which

- player 1 always proposes x^* and accepts y iff $y_1 \geq y_1^*$
- player 2 always proposes y^* and accepts x iff $x_2 \geq x_2^*$

where

$$x_1^* = \frac{1-\delta_2}{1-\delta_1\delta_2} \qquad \text{and} \qquad y_1^* = \frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2}$$

Proposition

The strategy pair s^* is the unique subgame perfect equilibrium of the bargaining game of alternating offers

Bargaining game of alternating offers: SPE

Proof that s^* is subgame perfect equilibrium

- In arbitrary infinite horizon game, strategy profiles that satisfy one-deviation property may not be subgame perfect equilibria
- But this property *does* hold in bargaining game of alternating offers (in which the single infinite history is the worst terminal history for each player)

Proposition

A strategy profile in the bargaining game of alternating offers is a subgame perfect equilibrium if and only if it satisfies the one-deviation property

Bargaining game of alternating offers: SPE

Proof that s^* is subgame perfect equilibrium

s_1^*: P1 always proposes x^* and accepts y iff $y_1 \geq y_1^* = \delta_1 x_1^*$
s_2^*: P2 always proposes y^* and accepts x iff $x_2 \geq x_2^* = \delta_2 y_2^*$

- Will show that s^* satisfies one-deviation property
- 2 types of subgame: first move offer, first move response

Subgame in which first move is offer

- Suppose offer is made by P1, and fix P2's strategy at s_2^*
- P1 uses $s_1^* \Rightarrow$ P1 proposes x^*, which P2 accepts $\Rightarrow$ P1's payoff is x_1^*
- P1 deviates from s_1^* in first period of subgame:
  - P1 offers P2 $> x_2^* \Rightarrow$ P2 accepts $\Rightarrow$ P1's payoff is $< x_1^*$
  - P1 offers P2 $< x_2^* \Rightarrow$ P2 rejects, P2 proposes $y^* \Rightarrow$ P1 accepts, obtaining payoff $\delta_1 y_1^*$
- $\delta_1 y_1^* = \delta_1^2 x_1^* < x_1^*$, so P1's proposing x^* is optimal
- Symmetric argument if first offer is made by P2

Bargaining game of alternating offers: SPE

Proof that s^* is subgame perfect equilibrium continued

s_1^*: P1 always proposes x^* and accepts $y \Leftrightarrow y_1 \geq y_1^* = \delta_1 x_1^*$
s_2^*: P2 always proposes y^* and accepts $x \Leftrightarrow x_2 \geq x_2^* = \delta_2 y_2^*$

Subgame in which first move is response to offer

- Suppose P1 is responding to y, and fix P2's strategy at s_2^*

  $y_1 \geq y_1^*$ P1 uses $s_1^* \Rightarrow$ she accepts offer $\Rightarrow$ payoff y_1
  P1 deviates from s_1^* in first period of subgame $\Rightarrow$ rejects offer $\Rightarrow$ P1 proposes $x^* \Rightarrow$ P2 accepts $\Rightarrow$ payoff $\delta_1 x_1^* = y_1^*$ for P1

  $y_1 < y_1^*$ P1 uses $s_1^* \Rightarrow$ she rejects offer and proposes $x^* \Rightarrow$ P2 accepts $\Rightarrow$ payoff $\delta_1 x_1^* = y_1^*$ for P1
  P1 deviates from s_1^* in first period of subgame $\Rightarrow$ accepts offer $\Rightarrow$ payoff $y_1 < y_1^*$

- So P1's response to offer is optimal
- Symmetric argument applies if responder is P2

Bargaining game of alternating offers: SPE

Conclusion

Strategy pair s^* defined by

- player 1 always proposes x^* and accepts $y \Leftrightarrow y_1 \geq y_1^*$
- player 2 always proposes y^* and accepts $x \Leftrightarrow x_2 \geq x_2^*$,

where

$$x_1^* = \frac{1-\delta_2}{1-\delta_1\delta_2} \quad \text{and} \quad y_1^* = \frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2}$$

(so that $y_1^* = \delta_1 x_1^*$ and $x_2^* = \delta_2 y_2^*$) satisfies one-deviation property and thus is a subgame perfect equilibrium

Defining property of SPE: each player indifferent between accepting and rejecting proposal made in equilibrium

Proof that s^* is *unique* subgame perfect equilibrium ($\Rightarrow$ no nonstationary SPE) is a little intricate (see book)

Bargaining game of alternating offers: properties of SPE

Efficiency

- Player 2 accepts player 1's first offer, so agreement is reached immediately; no resources are wasted in delay
- Intuition relates this feature to perfect information:
  - outcome not reached immediately $\Rightarrow$ alternative outcome that both players prefer (same outcome immediately)
  - given perfect information, players should perceive and pursue this alternative outcome?
- Nevertheless, some variants of model with perfect information have SPEs in which agreement is *not* reached immediately (see, e.g., Exercise 125.2b)

Bargaining game of alternating offers: properties of SPE

$$(x_1^*, x_2^*) = \left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2} \right)$$

Changes in patience

- Given δ_2, equilibrium payoff x_1^* of P1 increases as $\delta_1 \to 1$: given patience of P2, P1's share increases as she becomes more patient
- As P1 becomes extremely patient (δ_1 close to 1), her share approaches 1
- Symmetrically, fixing patience of P1, P2's share increases to 1 as she becomes more patient

Bargaining game of alternating offers: properties of SPE

$$(x_1^*, x_2^*) = \left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right)$$

First-mover advantage

- $\delta_1 = \delta_2 = \delta \Rightarrow$ only asymmetry in game is that P1 moves first

$\Rightarrow$ P1's equilibrium payoff is

$$\frac{1-\delta}{1-\delta^2} = \frac{1}{1+\delta} > \tfrac{1}{2}$$

Payoff $\to \frac{1}{2}$ as $\delta \to 1$

- Thus players equally and only slightly impatient $\Rightarrow$ P1's advantage small and outcome almost symmetric

Bargaining game of alternating offers

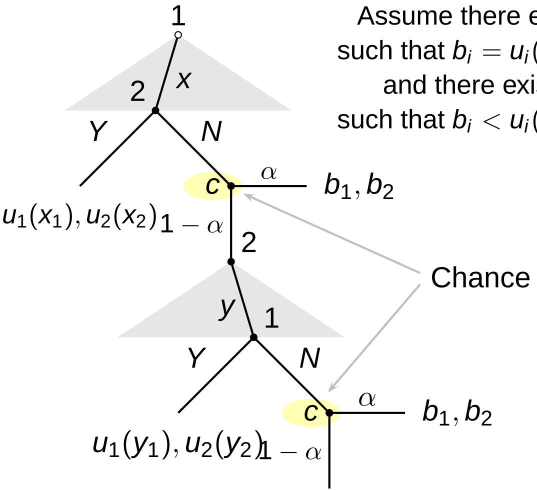
Many players

- Model can be extended to many players (e.g. all players have to agree to proposal)
- With more than two players, game has many SPEs: in fact, for *every* possible agreement there is an SPE in which that agreement is realized immediately
- Game has only one *stationary* equilibrium (any given player always makes same offer and uses same rule to respond to offers), though it is not clear that this equilibrium is right one to select

Bargaining game of alternating offers with risk of breakdown

- Motivation for players to reach agreement in bargaining game of alternating offers is their impatience
- Will now study variant of model in which motivation is possibility that exogenous event will cause bargaining to break down
- Possibility of breakdown is enough to induce agreement, so assume that discount factors are both 1

Bargaining game of alternating offers with risk of breakdown



Assume there exists (z_1, z_2) such that $b_i = u_i(z_i)$ for $i = 1, 2$ and there exists (x_1, x_2) such that $b_i < u_i(z_i)$ for $i = 1, 2$

Notes: Possibility of breakdown is *exogenous* (move of chance); no discounting; Bernoulli payoff functions u_1, u_2

Bargaining game of alternating offers with risk of breakdown

By logic similar to that for bargaining game of alternating offers, game has unique SPE

In this equilibrium,

- player 1 always proposes $\hat{x}(\alpha)$ and accepts $y \Leftrightarrow y_1 \geq \hat{y}_1(\alpha)$
- player 2 always proposes $\hat{y}(\alpha)$ and accepts $x \Leftrightarrow x_2 \geq \hat{x}_2(\alpha)$

and each player is indifferent between accepting and rejecting other player's equilibrium proposal, so that

$$u_1(\hat{y}_1(\alpha)) = (1-\alpha)u_1(\hat{x}_1(\alpha)) + \alpha b_1$$
$$u_2(\hat{x}_2(\alpha)) = (1-\alpha)u_2(\hat{y}_2(\alpha)) + \alpha b_2$$

Bargaining game of alternating offers with risk of breakdown: Risk neutral players

$$u_1(\hat{y}_1(\alpha)) = (1-\alpha)u_1(\hat{x}_1(\alpha)) + \alpha b_1$$
$$u_2(\hat{x}_2(\alpha)) = (1-\alpha)u_2(\hat{y}_2(\alpha)) + \alpha b_2$$

If $u_i(x_i) = x_i$ for each player i then

$$\hat{y}_1(\alpha) = (1-\alpha)\hat{x}_1(\alpha) + \alpha b_1$$
$$\hat{x}_2(\alpha) = (1-\alpha)\hat{y}_2(\alpha) + \alpha b_2$$

so that

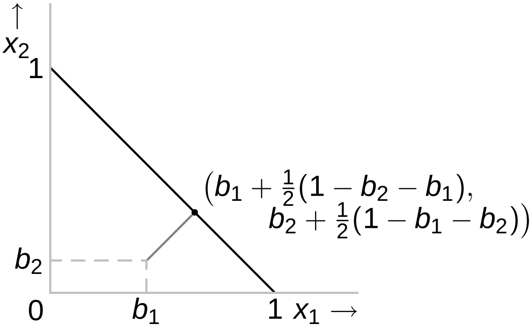
$$\hat{x}_1(\alpha) = \frac{1 - b_2 + (1-\alpha)b_1}{2-\alpha}, \quad \hat{y}_1(\alpha) = \frac{(1-\alpha)(1-b_2) + b_1}{2-\alpha}$$

Bargaining game of alternating offers with risk of breakdown: Risk neutral players

$$\hat{x}_1(\alpha) = \frac{1 - b_2 + (1-\alpha)b_1}{2-\alpha}, \qquad \hat{y}_1(\alpha) = \frac{(1-\alpha)(1-b_2) + b_1}{2-\alpha}$$

$$(\hat{x}_1(\alpha), \hat{x}_2(\alpha)) \xrightarrow[\alpha \to 0]{} \left(b_1 + \tfrac{1}{2}(1-(b_1+b_2)), b_2 + \tfrac{1}{2}(1-(b_1+b_2))\right)$$

$\Rightarrow$ "split the difference" ($1 - b_1 - b_2$ is surplus over breakdown outcome)



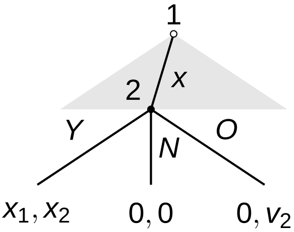
Bargaining game of alternating offers

Summary

- Bargaining game of alternating offers with discounting has unique SPE
- Agreement is reached immediately (efficient outcome)
- Player is more patient $\Rightarrow$ higher payoff
- Players equally patient, with δ close to 0 $\Rightarrow$ outcome close to $(\frac{1}{2}, \frac{1}{2})$
- With risk of breakdown (and no discounting), unique SPE with outcome close to "split the difference" solution over breakdown outcome when probability of breakdown close to 0

Outside options

- In model with risk of breakdown, bargaining breaks down independently of players' actions
- What if players can *choose* to terminate bargaining?
- One-period example ($v_2 > 0$):

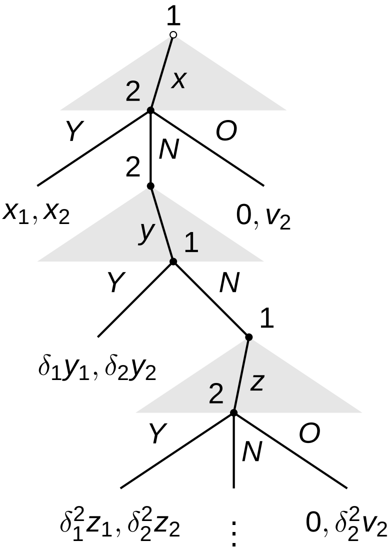


SPE	
$(1 - v_2, v_2)$	no optimal action
Y if $x_2 \geq v_2$ O if $x_2 < v_2$	Y if $x_2 > v_2$ O if $x_2 \leq v_2$

- Without outside option, P2 gets 0 $\Rightarrow$ outside option raises her SPE payoff to v_2

Outside options

Infinite horizon



Outside options: Subgame perfect equilibrium

Let

- $s^* =$ unique SPE of bargaining game of alternating offers with no outside options
- $x^*, y^* =$ proposals of players 1 and 2 in this equilibrium

Proposition

If $v_2 < x_2^*$ then s^* is unique SPE of infinite horizon game with outside option for player 2.
If $v_2 > x_2^*$ then the infinite horizon game with an outside option for player 2 has a unique SPE, in which

- player 1 always proposes $(1 - v_2, v_2)$ and accepts $y \Leftrightarrow y_1 \geq \delta_1(1 - v_2)$
- player 2 always proposes $(\delta_1(1 - v_2), 1 - \delta_1(1 - v_2))$ and accepts x if $x_2 \geq v_2$ and opts out otherwise.

Outside options: Subgame perfect equilibrium

Proposition

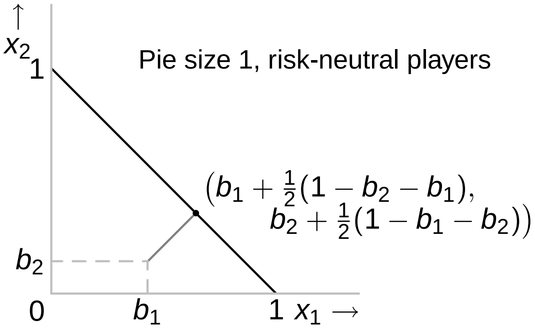
If $v_2 < x_2^*$ then s^* is unique SPE of infinite horizon game with outside option for player 2.
If $v_2 > x_2^*$ then the infinite horizon game with an outside option for player 2 has a unique SPE, in which

- player 1 always proposes $(1 - v_2, v_2)$ and accepts $y \Leftrightarrow y_1 \geq \delta_1(1 - v_2)$
- player 2 always proposes $(\delta_1(1 - v_2), 1 - \delta_1(1 - v_2))$ and accepts x if $x_2 \geq v_2$ and opts out otherwise.

- Result says that player's outside option affects SPE only if it is worth more than her equilibrium payoff in its absence
- Uniqueness is sensitive to timing of outside options
  - If player can opt out after opponent rejects offer, get multiple SPEs, with different properties

Outside options vs. exogenous breakdown

- In model with exogenous *risk of breakdown*, increase in player's breakdown payoff always increases her equilibrium payoff, even if breakdown payoff $<$ equilibrium payoff
  - Increase in breakdown payoff increases player's expected payoff in subgame following rejection of offer



Outside options vs. exogenous breakdown

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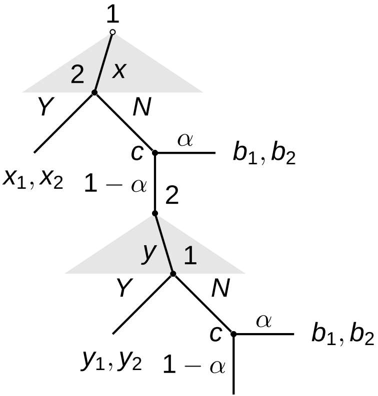

Outside options vs. exogenous breakdown

- In model with exogenous *risk of breakdown*, increase in player's breakdown payoff always increases her equilibrium payoff, even if breakdown payoff $<$ equilibrium payoff
  - Increase in breakdown payoff increases player's expected payoff in subgame following rejection of offer
- If breaking off negotiations is an *option*, it affects player's equilibrium payoff only if payoff it yields exceeds her equilibrium payoff in its absence
  - Player rationally takes option only when it benefits her, so option worse than equilibrium payoff in its absence is irrelevant