Economics 2030

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Solutions for Tutorial 3

1. (a) The argument is exactly the same as it is for an auction with no reserve price.

   (b) If $v_i < r$, then the expected price paid by i is 0 (the bidder never obtains the object).

   Now suppose that $v_i \geq r$. With probability r, the other player's valuation is less than r, in which case the player pays r, and with probability $1 - v_i$ the other player's valuation is greater than v_i, in which case the player does not win. In the remaining case, the other player's valuation is between r and v_i. This case occurs with probability $v_i - r$, and the other player's valuation is uniformly distributed between r and v_i. Thus the expected price paid by the player in this case is $\frac{1}{2}(r + v_i)$. Putting all these cases together, the expected price is

$$\begin{aligned}\Pr(v_j < r)r + \Pr(r < v_j < v_i)\tfrac{1}{2}(r + v_i) &= r^2 + \tfrac{1}{2}(v_i - r)(v_i + r) \\ &= \tfrac{1}{2}(v_i^2 + r^2).\end{aligned}$$

   (c) The expected revenue of the auctioneer is twice the expected value of $\pi(v_i)$. (Twice because there are two bidders.) The valuation v_i is uniformly distributed on $[0,1]$, and $\pi(v_i) = 0$ if $v_i < r$, so the expected value of $\pi(v_i)$ is

$$\int_r^1 \pi(v_i)\, dv_i.$$

   We have

$$\begin{aligned}\int_r^1 \pi(v_i)\, dv_i &= \tfrac{1}{2}\int_r^1 (v_i^2 + r^2)\, dv_i \\ &= \tfrac{1}{2}\left[\tfrac{1}{3}v_i^3 + r^2 v_i\right]_r^1 \\ &= \tfrac{1}{2}\left(\tfrac{1}{3} + r^2 - \tfrac{4}{3}r^3\right).\end{aligned}$$

   The maximizer of this function is the interior value of r for which its derivative is zero, which is $r = \frac{1}{2}$. (Sketch the function.)

   Thus the optimal reserve price is $\frac{1}{2}$.

2. (a) The game is defined as follows.

Players The n bidders.

States The set of all profiles $(v_1, \ldots, v_n)$ of valuations, where $v_i \in [0, 1]$ for all i.

Actions Each player's set of actions is the set of possible bids (nonnegative numbers).

Signals The set of signals that each player may observe is the set of possible valuations. The signal function τ_i of each player i is given by $\tau_i(v_1, \ldots, v_n) = v_i$ (each player knows her own valuation).

Prior beliefs Each player assigns probability $\Pi_{j=1}^n F(v_j)$ to the set of states in which the valuation of each player j is at most v_j, where F is the cumulative distribution function of the uniform distribution.

Preferences Player i's preferences are represented by the expected value of her Bernoulli payoff, which assigns to any pair $((b_1, \ldots, b_n), (v_1, \ldots, v_n))$ the payoff $v_i - b_i$ if b_i wins and $-b_i$ if it loses.

(b) Look for a symmetric equilibrium, in which each player bids $\beta(v)$ when her valuation is v, where β is increasing. In such an equilibrium, the expected payoff of a player with valuation v who bids b is

$$v(\beta^{-1}(b))^{n-1} - b.$$

An interior maximizer satisfies

$$(n-1)v(\beta^{-1}(b))^{n-2}/\beta'(\beta^{-1}(b)) - 1 = 0.$$

In an equilibrium, the bid $\beta(v)$ maximizes the payoff, so that if it is between 0 and 1 we have

$$(n-1)v^{n-1}/\beta'(v) = 1,$$

or

$$\beta'(v) = (n-1)v^{n-1}.$$

We conclude that

$$\beta(v) = (n-1)v^n/n + C.$$

where C is a constant. In an equilibrium, the bid of a player with valuation v is at most v (otherwise the player can increase her payoff by bidding 0), so $C = 0$.

We conclude that if the game has a symmetric equilibrium in which β is increasing, then $\beta(v) = (n-1)v^n/n$ for all v.

3. (a) Here is the game:

   Players $N = \{1,2\}$.

   States The set of pairs (c_1, c_2) where $0 \le c_i \le 1$ for $i = 1, 2$.

   Actions $A_1 = A_2 = \{W, N\}$.

   Signals $T_1 = T_2 = [0,1]$. $\tau_i(c_1, c_2) = c_i$ for $i = 1, 2$.

   Beliefs Each player's prior is uniform on $[0,1]^2$.

   Payoffs The payoff of player i is $\lambda(2-\lambda) - c_i$ if both players work, $\lambda - c_i$ is she works and the other player does not, λ if she does not work and the other player does, and 0 if neither player works.

   (b) A reasonable guess is that the game has a Nash equilibrium in which each player i works if and only if $c_i \le \bar{c}_i$, for some $\bar{c}_i$, $i = 1$, 2. For such a pair of strategies to be an equilibrium, player i of type $\bar{c}_i$ must obtain the same expected payoff from working as from not working. The payoff of player 1 of type $\bar{c}_1$ from working is

$$(\lambda(2-\lambda) - \bar{c}_1)\Pr(c_2 \le \bar{c}_2) + (\lambda - \bar{c}_1)\Pr(c_2 > \bar{c}_2)$$

or

$$(\lambda(2-\lambda) - \bar{c}_1)\bar{c}_2 + (\lambda - \bar{c}_1)(1 - \bar{c}_2)$$

and her payoff from not working is

$$\lambda \Pr(c_2 \le \bar{c}_2) = \lambda \bar{c}_2.$$

For these payoffs to be equal, we need

$$\bar{c}_1 = \lambda(1 - \lambda \bar{c}_2).$$

A similar argument for player 2 leads to the requirement

$$\bar{c}_2 = \lambda(1 - \lambda \bar{c}_1).$$

These two conditions together imply that $\bar{c}_1 = \bar{c}_2 = \lambda/(1+\lambda^2)$. To check that the strategy pair is an equilibrium we need to show, for $i = 1$, 2, that types of player i less than $\lambda/(1+\lambda^2)$ prefer not to work and types greater than $\lambda/(1+\lambda^2)$ prefer to work, given that types of player j less than $\bar{c}_j$ work and types greater than $\bar{c}_j$

do not work, for $j \neq i$. The payoff to type c_i of player i in this case is $\lambda\bar{c}_j$ if she does not work and $(\lambda(2-\lambda)-c_i)\bar{c}_j + (\lambda - c_i)(1-\bar{c}_2)$ if she works. The former is independent of c_i whereas the latter is decreasing in c_i, so indeed types less than $\bar{c}_i$ prefer not to work and ones greater than $\bar{c}_i$ prefer to work. Thus the strategy pair is a Nash equilibrium.

4. No.

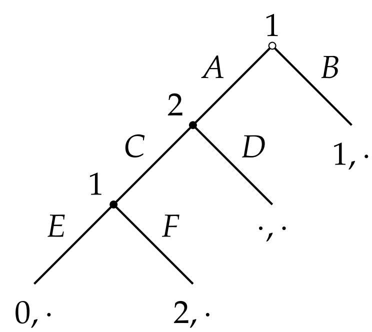

Figure 1. An extensive game with a strategy profile that satisfies the "one deviation property along the equilibrium path" but is not a Nash equilibrium.

In the game in Figure 1 the strategy pair (BE, C) is not a Nash equilibrium but satisfies the property.