Economics 2030

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Solutions to Problem Set 4

1. We have

$$b_1(q_L, q_H) = \begin{cases} \frac{1}{2}(\alpha - c - (\theta q_L + (1-\theta)q_H)) & \text{if } \theta q_L + (1-\theta)q_H \le \alpha - c \\ 0 & \text{otherwise.} \end{cases}$$

The best response function of each type of player 2 is similar:

$$b_I(q_1) = \begin{cases} \frac{1}{2}(\alpha - c_I - q_1) & \text{if } q_1 \le \alpha - c_I \\ 0 & \text{otherwise} \end{cases}$$

for $I = L, H$.

The three equations that define a Nash equilibrium are

$$q_1^* = b_1(q_L^*, q_H^*),\ q_L^* = b_L(q_1^*), \text{ and } q_H^* = b_H(q_1^*).$$

Solving these equations under the assumption that they have a solution in which all three outputs are positive, we obtain

$$\begin{aligned} q_1^* &= \tfrac{1}{3}(\alpha - 2c + \theta c_L + (1-\theta)c_H) \\ q_L^* &= \tfrac{1}{3}(\alpha - 2c_L + c) - \tfrac{1}{6}(1-\theta)(c_H - c_L) \\ q_H^* &= \tfrac{1}{3}(\alpha - 2c_H + c) + \tfrac{1}{6}\theta(c_H - c_L). \end{aligned}$$

If both firms know that the unit costs of the two firms are c_1 and c_2 then in a Nash equilibrium the output of firm i is $\frac{1}{3}(\alpha - 2c_i + c_j)$. (Check that by computing the best response functions in that case.) In the case of imperfect information considered here, firm 2's output is less than $\frac{1}{3}(\alpha - 2c_L + c)$ if its cost is c_L and is greater than $\frac{1}{3}(\alpha - 2c_H + c)$ if its cost is c_H. Intuitively, the reason is as follows. If firm 1 knew that firm 2's cost were high then it would produce a relatively large output; if it knew this cost were low then it would produce a relatively small output. Given that it does not know whether the cost is high or low it produces a moderate output, less than it would if it knew firm 2's cost were high. Thus if firm 2's cost is in fact high, firm 2 benefits from firm 1's lack of knowledge and optimally produces more than it would if firm 1 knew its cost.

2. (a) The following Bayesian game models the situation.

   Players The two people.

   States The set of pairs (θ_1, θ_2) where $0 \leq \theta_i \leq 1$ for $i = 1, 2$.

   Actions The actions of each player are *contribute* (C) and *don't contribute* (D).

   Signals The set of signals that each player i may observe is $[0, 1]$ and her signal function is defined by $\tau_i(\theta_1, \theta_2) = \theta_i$ for all (θ_1, θ_2).

   Prior belief The prior belief of each player is that each θ_i is independently uniform on $[0, 1]$.

   Preferences The preferences of each player i are represented by the payoff function

$$u_i((a_1, a_2), (\theta_1, \theta_2)) = \begin{cases} (\theta_i)^2 - c & \text{if } a_i = C \\ (\theta_i)^2 & \text{if } a_i = D \text{ and } a_j = C \\ 0 & \text{otherwise.} \end{cases}$$

   (b) One possible guess that the game has an equilibrium in which each player i contributes if and only if $\theta_i \geq \overline{\theta}$ for some value of $\overline{\theta}$. To be a Nash equilibrium, such a strategy pair must have the property that a player with $\theta_i = \overline{\theta}$ is indifferent between C and D, which requires

$$(\overline{\theta})^2 - c = \Pr(\theta \geq \overline{\theta})(\overline{\theta})^2,$$

   or

$$(\overline{\theta})^2 - c = (1 - \overline{\theta})(\overline{\theta})^2,$$

   which implies that $\overline{\theta} = c^{1/3}$.

   We can now verify that the strategy pair in which each player i contributes if and only if $\theta_i \geq c^{1/3}$ is indeed a Nash equilibrium. If $\theta_i \geq c^{1/3}$ and player i deviates from the strategy pair by switching to D, her expected payoff changes from $(\theta_i)^2 - c$ to $(1 - c^{1/3})(\theta_i)^2$, which is at most $(\theta_i)^2 - c$ given $\theta_i \geq c^{1/3}$. Symmetrically, if $\theta_i < c^{1/3}$ then player i is worse off switching from D to C.

3. The following Bayesian game models the situation.

Players The two people.

States The set of states is $\{strong, weak\}$.

Actions The set of actions of each player is $\{fight, yield\}$.

Prior beliefs The prior belief of each player i is $p_i(strong) = \alpha$, $p_i(weak) = 1 - \alpha$.

Signals Player 1 receives the same signal in each state, whereas player 2 receives different signals in the two states.

Payoffs The players' Bernoulli payoffs are shown in the figure in the problem.

State: *strong*

	F	Y
F	$-1, 1^*$	$1, 0$
Y	$0, 1^*$	$0, 0$

State: *weak*

	F	Y
F	$1, -1$	$1, 0^*$
Y	$0, 1^*$	$0, 0$

Figure 1. The player's Bernoulli payoff functions in Exercise 3. The asterisks indicate the best responses of each type of player 2.

The best responses of each type of player 2 are indicated by asterisks in Figure 1. Thus if $\alpha < \frac{1}{2}$ then player 1's best action is *fight*, whereas if $\alpha > \frac{1}{2}$ her best action is *yield*. Hence

- if $\alpha < \frac{1}{2}$ the game has a unique Nash equilibrium, in which player 1 chooses *fight* and player 2 chooses *fight* if she is strong and *yield* if she is weak
- if $\alpha > \frac{1}{2}$ the game has a unique Nash equilibrium, in which player 1 chooses *yield* and player 2 chooses *fight* whether she is strong or weak.

4. The game is defined as follows.

Players Firms A and T.

States The set of possible values of firm T (the integers from 0 to 100).

Actions Firm A's set of actions is its set of possible bids (nonnegative numbers), and firm T's set of actions is the set of possible cutoffs (nonnegative numbers) above which it will accept A's offer.

Signals Firm A receives the same signal in every state; firm T receives a different signal in every state.

Beliefs The single type of firm A assigns an equal probability to each state; each type of firm T assigns probability 1 to the single state consistent with its signal.

Payoff functions If firm A bids y, firm T's cutoff is at most y, and the state is x, then A's payoff is $\frac{3}{2}x - y$ and T's payoff is y. If firm A bids y, firm T's cutoff is greater than y, and the state is x, then A's payoff is 0 and T's payoff is x.

To find the Nash equilibria of this game, first consider the behavior of each type x of firm T. Type x is at least as well off accepting the offer y than it is rejecting it if and only if $y \geq x$. Thus any best response of type x to an offer y has a cutoff of at most y if $y > x$ and a cutoff of greater than y if $y < x$.

Now consider firm A. If it bids y then each type x of T with $x < y$ accepts its offer, and each type x of T with $x > y$ rejects the offer. Thus the expected value of the types that accept an offer $y \leq 100$ is $\frac{1}{2}q(y)$, where $q(y)$ is the largest integer at most equal to y, and the expected value of the types that accept an offer $y > 100$ is 50. If the offer y is accepted then A's payoff is $\frac{3}{2}x - y$, so that its expected payoff is $\frac{3}{2}(\frac{1}{2}q(y)) - y$ if $y \leq 100$ and $\frac{3}{2}(50) - y = 75 - y$ if $y > 100$. In both cases this expected payoff is negative. (In the first case it is approximately $\frac{1}{4}y$.) Thus firm A's optimal bid is 0!

We conclude that a strategy pair is a Nash equilibrium of the game if and only if firm A bids 0 and the cutoff for accepting an offer for each type x of firm T is greater than 0 if $x > 0$ and at least 0 if $x = 0$.

Even though firm A can increase firm T's value, it is not willing to make a positive bid in equilibrium because firm T's interest is in accepting only offers that exceed its value, so that the average type that accepts an offer has a value of only half the offer. As A decreases its offer, the value of the average firm that accepts the offer decreases: the selection of firms that accept the offer is adverse to A's interest.

5. (a) In state γ, player 1 knows player 2's preferences, because she knows that the state is either β or γ, and player 2's preferences are the same in both of those states.

   Player 2 also knows player 1's preferences, because she knows the state is γ.

   Player 2 also knows that player 1 knows player 2's preferences: player 2's preferences are the same in all three states.

Player 1 does not know that player 2 knows player 1's preferences: player 1 knows that the state is either β or γ, and in state β player 2 does not know whether the state is α or β; player 1's preferences in α and β differ.

(b) In any Nash equilibrium, the action of player 1 when she receives the signal $\tau_1(\alpha)$ is R, because R strictly dominates L.

Suppose player 2's signal is $\tau_2(\alpha) = \tau_2(\beta)$. Then her best action is R, *regardless of player 1's action in state* β:

- If player 1 chooses L in state β then player 2's expected payoff to L is $\frac{3}{4} \cdot 0 + \frac{1}{4} \cdot 2 = \frac{1}{2}$, and her expected payoff to R is $\frac{3}{4} \cdot 1 + \frac{1}{4} \cdot 0 = \frac{3}{4}$.
- If player 1 chooses R in state β then player 2's expected payoff to L is 0, and her expected payoff to R is 1.
- Thus in any Nash equilibrium player 2's action when her signal is $\tau_2(\alpha) = \tau_2(\beta)$ is R.

Now suppose player 1's signal is $\tau_1(\beta) = \tau_1(\gamma)$. By same argument as before, 1's best action is R, regardless of player 2's action in state γ. Thus in any Nash equilibrium player 1's action in this case is R.

Finally, given that player 1's action in state γ is R, player 2's best action in this state is also R.

Hence the unique Nash equilibrium is $((R, R), (R, R))$.

(c) In state δ, player 1 knows player 2's preferences, player 2 knows player 1's preferences, player 2 knows that player 1 knows player 2's preferences, and player 1 knows that player 2 knows player 1's preferences.

The unique Nash equilibrium of the game is $((R, R, R), (R, R))$.