Economics 2030

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Solutions to Problem Set 3

1. For player 2 the action R is strictly dominated by the mixed strategy that assigns probability $\frac{1}{4}$ to L and probability $\frac{3}{4}$ to M. (It is strictly dominated by other strategies too.) Thus in every Nash equilibrium player 2 assigns probability 0 to R, and hence we can eliminate R from consideration.

   We can find the Nash equilibria of the resulting game (in which player 1's actions are T, M, and B and player 2's actions are L and M) by considering each possible pair of supports for the strategies in turn.

   One action for each player By inspection, the only cases that yield Nash equilibria are (T, L) and (M, M).

   Two actions for player 1, one for player 2 In no case is player 1 indifferent between two actions, given the action of player 2, so in no case is there an equilibrium.

   One action for player 1, two for player 2 In no case is player 2 indifferent between two actions, given the action of player 1, so in no case is there an equilibrium.

   (T, M) **for player 1,** (L, M) **for player 2** Denote by p the probability player 1 assigns to T and by q the probability player 2 assigns to L. For player 1 to be indifferent between T and M we need $q = \frac{1}{2}$. For player 2 to be indifferent between L and R we need $p = \frac{4}{5}$. Given $q = \frac{1}{2}$, player 1's expected payoff to B is the same as her expected payoff to T and M, so this pair of strategies is a Nash equilibrium.

   (T, B) **for player 1,** (L, M) **for player 2** Given the player 1 assigns positive probability only to T and B, player 2's payoff to L exceeds her payoff to R. Thus there is no equilibrium with these supports.

(M, B) **for player 1,** (L, M) **for player 2** Denote by p the probability player 1 assigns to M and by q the probability player 2 assigns to L. For player 1 to be indifferent between M and B we need $q = \frac{1}{2}$. For player 2 to be indifferent between L and R we need $p = \frac{1}{5}$. Given $q = \frac{1}{2}$, player 1's expected payoff to B is the same as her expected payoff to T and M, so this pair of strategies is a Nash equilibrium.

(T, M, B) **for player 1,** (L, M) **for player 2** Denote by p the probability player 1 assigns to T and r the probability she assigns to M; denote by q the probability player 2 assigns to L. For player 1 to be indifferent between T, M, and B we need $q = \frac{1}{2}$. For player 2 to be indifferent between L and R we need $p + 2(1 - p - r) = 4r + 1 - p - r$ or $r = \frac{1}{5}$.

The equilibria in the fourth and sixth cases are special cases of the equilibria in the last case, so we conclude that the Nash equilibria of the game are

- $((1, 0, 0), (1, 0, 0))$
- $((0, 1, 0), (0, 1, 0))$
- any pair $((p_1, \frac{1}{5}, p_3), (\frac{1}{2}, \frac{1}{2}, 0))$ with $p_1 + p_3 = \frac{4}{5}$ and $p_1 \geq 0$, $p_3 \geq 0$.

2. (a) The game has no pure strategy Nash equilibrium and no mixed strategy Nash equilibrium in which one of the player's strategies is pure. Consider a mixed strategy Nash equilibrium in which neither player's strategy is pure. Denote the probability player 1 assigns to A by p and the probability player 2 assigns to A by q.

    For an equilibrium we need player 1's expected payoffs to A and B to be the same, or

$$(1 - q)v_A = qv_B + (1 - q)\pi v_B,$$

    which means that

$$q = 1 - v_B/[v_A + (1 - \pi)v_B] = (v_A - \pi v_B)/(v_A + (1 - \pi)v_B)$$

    (which is nonnegative and at most 1). Denote this probability by q^*. We need also player 2's expected payoffs to A and B to be the same, or

$$-(1 - p)v_B = -pv_A - (1 - p)\pi v_B,$$

which means that

$$p = 1 - v_A/[v_A + (1-\pi)v_B] = (1-\pi)v_B/(v_A + (1-\pi)v_B)$$

(which is nonnegative and at most 1). Denote this probability by p^*. Thus the game has a unique Nash equilibrium $((p^*, 1 - p^*), (q^*, 1 - q^*))$.

(b) Player 1's expected payoff in the equilibrium of the game in part *a* is $(1-q)v_A$, where q is the equilibrium probability that player 2 chooses A, and is thus equal to $v_A v_B/[v_A + (1-\pi)v_B]$.

Thus if $h \leq v_A v_B/[v_A + (1-\pi)v_B]$, a Nash equilibrium of the game is the mixed strategy equilibrium $((p^*, 1-p^*, 0), (q^*, 1-q^*))$.

If $h > v_A v_B/[v_A + (1-\pi)v_B]$, then for player 2's strategy $(q, 1-q)$, player 1's expected payoff to A is $(1-q)v_A$, her expected payoff to B is $qv_B + (1-q)\pi v_B$, and her expected payoff to C is h. Thus the strategy pair $((0,0,1),(q,1-q))$ is a Nash equilibrium if $(1-q)v_A \leq h$ and $qv_B + (1-q)\pi v_B \leq h$, or $1 - h/v_A \leq q \leq (h - \pi v_B)/[(1-\pi)v_B]$ (one of these equilibria is $((0,0,1),(q^*,1-q^*))$). (If $h \geq v_A$ then $((0,0,1),(0,1))$ is also a Nash equilibrium.)

3. (*a*) True. Suppose that the mixed strategy α_i' assigns positive probability to the action a_i', which is strictly dominated by the action a_i. Then $u_i(a_i, a_{-i}) > u_i(a_i', a_{-i})$ for all a_{-i}. Let α_i be the mixed strategy that differs from α_i' only in the weight that α_i' assigns to a_i' is transferred to a_i. That is, α_i is defined by $\alpha_i(a_i') = 0$, $\alpha_i(a_i) = \alpha_i'(a_i') + \alpha_i'(a_i)$, and $\alpha_i(b_i) = \alpha_i'(b_i)$ for every other action b_i. Then α_i strictly dominates α_i': for any a_{-i} we have $U(\alpha_i, a_{-i}) - U(\alpha_i', a_{-i}) = \alpha_i'(a_i')(u(a_i, a_{-i}) - u_i(a_i', a_{-i})) > 0$.

(*b*) False. Consider the game in Figure 1. Then player 1's mixed strategy that assigns probability $\frac{1}{2}$ to M and probability $\frac{1}{2}$ to B is strictly dominated by T, even though neither M nor B is strictly dominated.

	L	R
T	2	2
M	3	0
B	0	3

Figure 1. Player 1's payoffs in a strategic game with vNM preferences.

4. I look for a value of p such that the condition in Lemma 33.2 is satisfied.

   First consider a supporter of candidate A. If she votes then candidate A ties if all $k-1$ of her comrades vote, an event with probability p^{k-1}, and otherwise candidate A loses. Thus her expected payoff is
$$p^{k-1} - c.$$
   If she abstains, then candidate A surely loses, so her payoff is 0. Thus in an equilibrium in which $0 < p < 1$ the condition in Lemma 33.2 implies that $p^{k-1} = c$, or
$$p = c^{1/(k-1)}.$$

   Now consider a supporter of candidate B who votes. With probability p^k all of the supporters of candidate A vote, in which case the election is a tie; with probability $1 - p^k$ at least one of the supporters of candidate A does not vote, in which case candidate B wins. Thus the expected payoff of a supporter of candidate B who votes is
$$p^k + 2(1 - p^k) - c.$$

   If the supporter of candidate B switches to abstaining, then

   - candidate B loses if all supporters of candidate A vote, an event with probability p^k
   - candidate B ties if exactly $k-1$ supporters of candidate A vote, an event with probability $kp^{k-1}(1-p)$
   - candidate B wins if fewer than $k-1$ supporters of candidate A vote, an event with probability $1 - p^k - kp^{k-1}(1-p)$.

   Thus a supporter of candidate B who switches from voting to abstaining obtains an expected payoff of
$$kp^{k-1}(1-p) + 2(1 - p^k - kp^{k-1}(1-p)) = 2 - (2-k)p^k - kp^{k-1}.$$

   Hence in order for it to be optimal for such a citizen to vote (i.e. in order for the condition in Lemma 33.2 to be satisfied), we need
$$p^k + 2(1 - p^k) - c \geq 2 - (2-k)p^k - kp^{k-1},$$

or

$$kp^{k-1}(1-p) + p^k \geq c.$$

Finally, consider a supporter of candidate B who abstains. With probability p^k all the supporters of candidate A vote, in which case the candidates tie; with probability $1 - p^k$ at least one of the supporters of candidate A does not vote, in which case candidate B wins. Thus the expected payoff of a supporter of candidate B who abstains is

$$p^k + 2(1 - p^k).$$

If this citizen instead votes, candidate B surely wins (she gets $k + 1$ votes, while candidate A gets at most k). Thus the citizen's expected payoff is

$$2 - c.$$

Hence in order for the citizen to wish to abstain, we need

$$p^k + 2(1 - p^k) \geq 2 - c$$

or

$$c \geq p^k.$$

In summary, for equilibrium we need $p = c^{1/(k-1)}$ and

$$p^k \leq c \leq kp^{k-1}(1-p) + p^k.$$

Given $p = c^{1/(k-1)}$, $c = p^{k-1}$, so that the two inequalities are satisfied. Thus $p = c^{1/(k-1)}$ defines an equilibrium.

As c increases, the probability p, and hence the expected number of voters, *increases*.

5. (a) Every output greater than q^m is strictly dominated by q^m.

   (b) After all outputs greater than q^m are deleted, every output less than $q^m/2$ is strictly dominated by $q^m/2$.

The process can be continued; the only pair of outputs that survives all rounds is the Nash equilibrium.