ECO2030: Microeconomic Theory II, module 1

Lecture 6

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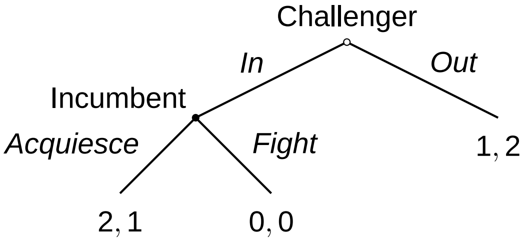
Extensive games with perfect information

- Strategic game is not natural model of situation in which actions are chosen sequentially
- Actions in strategic game can capture behavior that will unfold over time, but strategic game does not allow reevaluation of choices
- Model that explicitly captures sequential choices: *extensive game*

Example: entry game



- Two players, Challenger and Incumbent
- Small circle denotes start of game
- First, Challenger chooses *In* or *Out*
- If Challenger chooses *In*, Incumbent chooses *Acquiesce* or *Fight*
- If Challenger chooses *Out*, game ends
- Payoffs are numbers at bottom (challenger's payoff first)

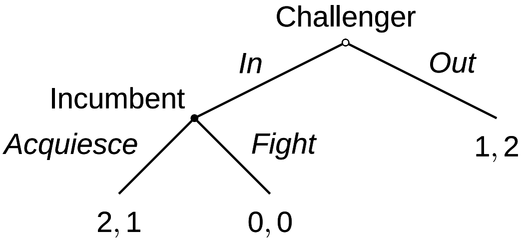
Extensive games with perfect information

Defined by

- set of players
- set of possible sequences of actions—*histories*
- specification of player who moves after any given history
- players' preferences over outcomes

Histories

- A history is a sequence of actions beginning at start of game
- Example: in entry game, the histories are $\varnothing$ (start of game), *In*, *Out*, (*In*, *Acquiesce*), and (*In*, *Fight*)



Histories

A set H of sequences is a set of histories if

- $\varnothing \in H$ (one possible history is null history—start of game)
- if $(a^1, \ldots, a^L, \ldots, a^K) \in H$ then $(a^1, \ldots, a^L) \in H$
- if $(a^1, \ldots, a^L) \in H$ for *every* positive L then $(a^1, \ldots) \in H$

A history h is terminal if

- h is infinite *or*
- $h = (a^1, \ldots, a^L)$ for some L and there is no a for which $(a^1, \ldots, a^L, a) \in H$

For set of histories H, denote set of terminal histories $Z(H)$

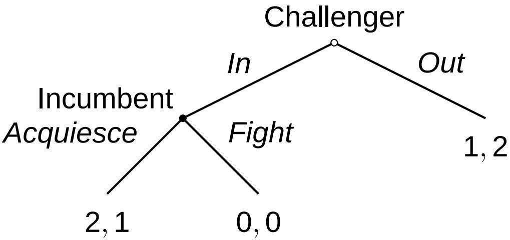
Extensive games with perfect information

Definition

An extensive game with perfect information consists of

- a set N (the set of *players*)
- a set H of histories
- a function $P : H \setminus Z(H) \to N$ (the *player function*, specifying the player who moves after each nonterminal history)
- for each player $i \in N$, a preference relation over $Z(H)$

Example



Players $N = \{\text{Challenger}, \text{Incumbent}\}$

Histories $H = \{\varnothing, In, Out, (In, Fight), (In, Acquiesce)\}$

Player function $P(\varnothing) = \text{Challenger}$, $P(In) = \text{Incumbent}$

Preferences $(In, Acquiesce) \succ_C Out \succ_C (In, Fight)$,
$Out \succ_I (In, Acquiesce) \succ_I (In, Fight)$

Actions

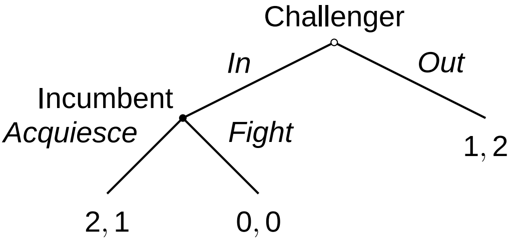
- Actions available to players when they move aren't explicit in definition
- Actions are defined implicitly:

  if i moves after the history $(a^1, \ldots, a^L)$ then her set of actions at this history is the set of values of a_i for which $(a^1, \ldots, a^L, a_i)$ is a history.

- More precisely, set of actions available to player who moves after history h is

$$A(h) = \{a : (h, a) \in H\}.$$

Example



- $A(\varnothing) = \{a : a \in H\} = \{In, Out\}$
- $A(In) = \{a : (In, a) \in H\} = \{Acquiesce, Fight\}$

Finite games

Finite horizon game
Game has finite horizon if every history is finite

Finite game
Game is finite if number of histories is finite

Strategy

Key concept!

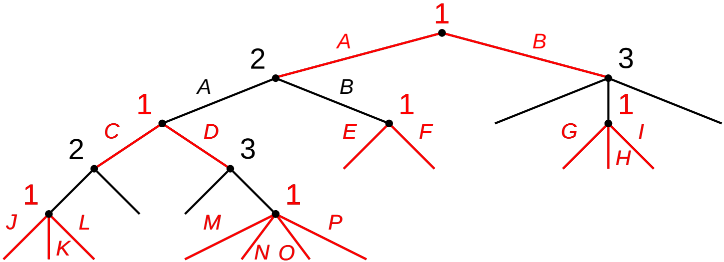
Definition

A strategy of player i in an extensive game with perfect information $\langle N, H, P, (\succsim_i)_{i \in N} \rangle$ is a function that assigns an action in $A(h)$ to **EVERY** nonterminal history $h \in H \setminus Z(H)$ for which $P(h) = i$

Finding player's set of strategies is mechanical:

- Make list of *all* histories after which player moves
- Player's set of strategies is set of all possible combinations of actions after these histories
- If player moves after k histories and has m_1 moves after first history, m_2 moves after second history, ..., m_k moves after kth history, total number of her strategies is $m_1 m_2 \cdots m_k$

Strategies: example

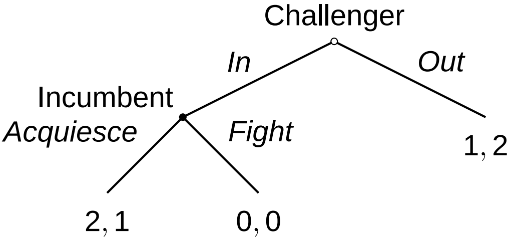


- Number of strategies of player 1:

$$2 \times 2 \times 2 \times 3 \times 3 \times 4 = 288$$

- *One* strategy: *ACEGJM*
- Let's look at some simpler examples ...

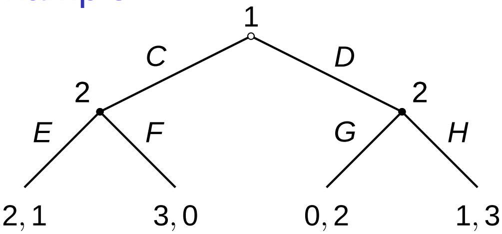
Strategies: Example



Challenger Moves only after null history. Two actions after this history, so two strategies: *In*, *Out*

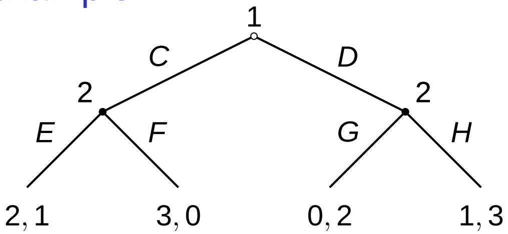
Incumbent Moves only after history *In*. Two actions after this history, so two strategies: *Acquiesce*, *Fight*

Strategies: example



Player 1 Moves only after null history. Two actions after this history, so two strategies: C, D

Strategies: example



Player 2 Moves after two histories:

C: two actions, E and F

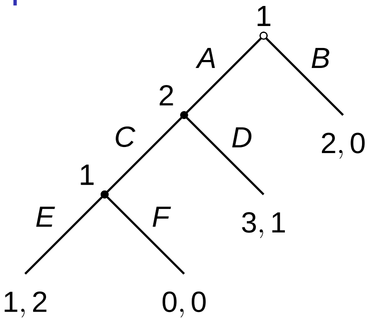
D: two actions, G and H

Hence four strategies:

- $s_2(C) = E$ and $s_2(D) = G$ (EG for short)
- $s_2(C) = E$ and $s_2(D) = H$ (EH for short)
- $s_2(C) = F$ and $s_2(D) = G$ (FG for short)
- $s_2(C) = F$ and $s_2(D) = H$ (FH for short)

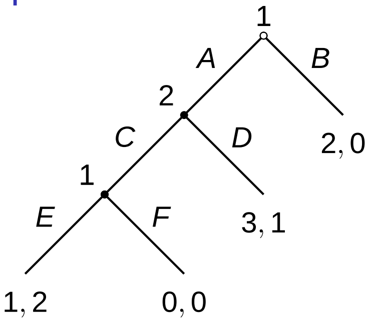
Strategy of player 2 in this game is *plan of action*

Strategies: example



Player 2 Moves after *one* history, A, and has 2 actions, C and D, so 2 strategies: C, D

Strategies: example

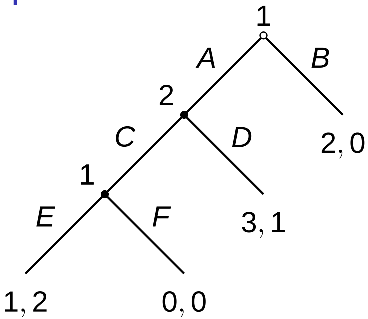


Player 1 Moves after

- null history: 2 actions, A and B
- history (A, C): 2 actions, E and F

So 4 strategies: AE, AF, BE, BF

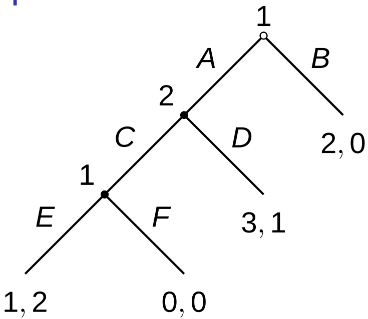
Strategies: example



Note

- Each strategy of player 1 specifies action after history (A, C) *even if it specifies B at beginning of game*!
- In general: definition of strategy requires action to be specified for *every* history after which it is player's turn to move, *even histories not reached if strategy is followed*

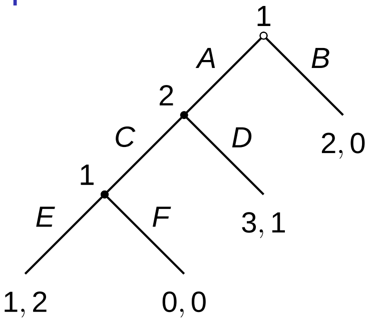
Strategies: example



One interpretation of strategy *BE* of player 1:

1. Action *E* models behavior of player 1 if, by chance, she doesn't choose *B* at start of game (though she intends to)

Strategies: example



Another interpretation of strategy BE of player 1:

2. When choosing between A and B,
   - player 1 has to think about action player 2 intends to take
   - player 1 knows that player 2's action depends on action player 2 thinks player 1 will take after history (A, C)

   Component E of player 1's strategy is her belief about player 2's belief about player 1's action after history (A, C)

Strategic form of extensive game

Given any extensive game, can now define strategic game

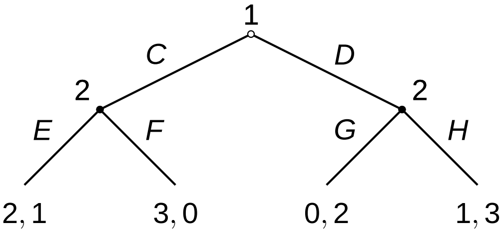
- Players: players in extensive game
- Actions of player i: *strategies* of player i in extensive game
- Players' payoffs to action profile: payoffs to terminal history that results when the players follow their strategies

Resulting strategic game is strategic form of extensive game

$$\begin{array}{ccc} \text{Extensive game} & & \text{Strategic form} \\ \langle N, H, P, (\succsim_i)_{i\in N}\rangle & \rightarrow & \langle N, (S_i)_{i\in N}, (\succsim_i^*)\rangle \end{array}$$

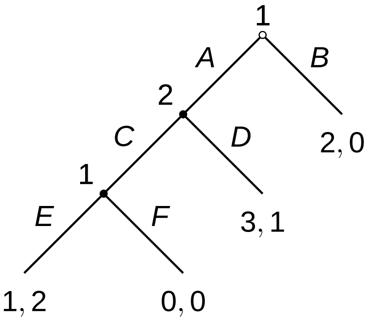
where S_i is set of strategies of player i in extensive game and $\succsim_i^*$ are i's preferences over strategy profiles induced by $\succsim_i$

Example of strategic form



	EG	EH	FG	FH
C	2, 1	2, 1	3, 0	3, 0
D	0, 2	1, 3	0, 2	1, 3

Example of strategic form



	C	D
AE	1, 2	3, 1
AF	0, 0	3, 1
BE	2, 0	2, 0
BF	2, 0	2, 0

Note duplicate strategies of player 1

Reduced strategic form:

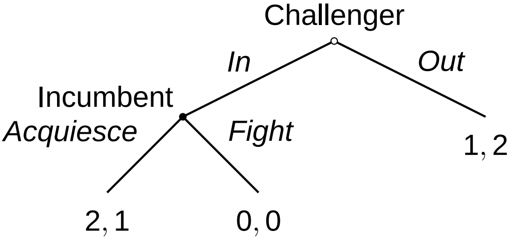
	C	D
AE	1, 2	3, 1
AF	0, 0	3, 1
X	2, 0	2, 0

Nash equilibrium

Definition

A Nash equilibrium of an extensive game with perfect information is a Nash equilibrium of its strategic form

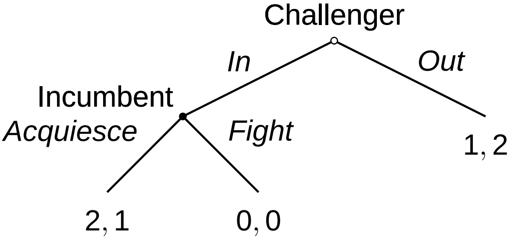
Example



	Acquiesce	*Fight*
In	2, 1	0, 0
Out	1, 2	1, 2

Nash equilibria: $(In, Acquiesce)$ and $(Out, Fight)$

Nash equilibrium: example



Nash equilibria

$(In, Acquiesce)$ Both actions played in equilibrium; each is optimal when played

$(Out, Fight)$ *Out* played in equilibrium, but *Fight* not played

- *Fight* is optimal given player 1 chooses *Out* (action of player 2 doesn't affect outcome)
- But *Fight* is not optimal if history *In* occurs
  - *Fight* can be interpreted as *non-credible threat*

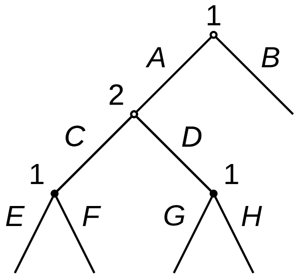
Nash equilibrium

- In Nash equilibrium, each player's strategy is optimal given other players' strategies
  $\Rightarrow$ each player's strategy optimal at *start* of game
- But a player's Nash equilibrium strategy may not be optimal in subgames not reached if players follow their strategies
- Notion of *subgame perfect equilibrium* requires that each player's strategy be optimal after *every* history, even histories that do not occur if every player follows her strategy

Subgames

For any nonterminal history h, subgame following h is part of game remaining once h has occurred
$\Rightarrow$ number of subgames $=$ number of nonterminal histories

Example



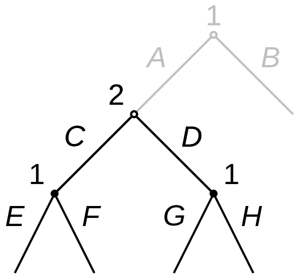
Subgame following $\varnothing$ (whole game)

Subgames

For any nonterminal history h, subgame following h is part of game remaining once h has occurred

$\Rightarrow$ number of subgames $=$ number of nonterminal histories

Example

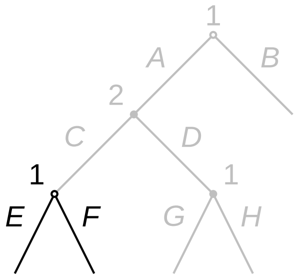


Subgame following A

Subgames

For any nonterminal history h, subgame following h is part of game remaining once h has occurred
$\Rightarrow$ number of subgames $=$ number of nonterminal histories

Example



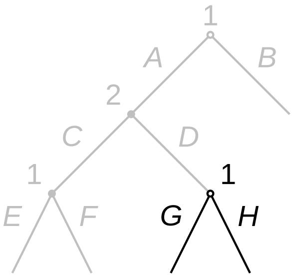
Subgame following (A, C)

Subgames

For any nonterminal history h, subgame following h is part of game remaining once h has occurred

$\Rightarrow$ number of subgames $=$ number of nonterminal histories

Example



Subgame following (A, D)

Nash equilibrium and subgame perfect equilibrium

- Nash equilibrium of extensive game previously defined as Nash equilibrium of its strategic form
- Can define it directly in terms of extensive game
- Let $\Gamma = \langle N, H, P, (\succsim_i) \rangle$ be extensive game with perfect information
- Let O be *outcome function* of Γ: $O(s) =$ terminal history that occurs when players use strategy profile s

Definition

A Nash equilibrium of $\langle N, H, P, (\succsim_i) \rangle$ is a strategy profile s^* such that for every player $i \in N$,

$$O(s^*_{-i}, s^*_i) \succsim_i O(s^*_{-i}, s_i) \text{ for every strategy } s_i \text{ of player } i$$

Subgame perfect equilibrium

- For any nonterminal history h, let
  - $\Gamma(h)$ = subgame of Γ following h
  - O_h = outcome function of $\Gamma(h)$
- For any strategy s_i of player i, let
  - $s_i|_h$ = strategy of player i defined by s_i in $\Gamma(h)$.

Definition

A subgame perfect equilibrium of $\langle N, H, P, (\succsim_i)\rangle$ is a strategy profile s^* such that for every player $i \in N$ and every nonterminal history $h \in H \setminus Z(H)$ for which $P(h) = i$,

$$O_h(s^*_{-i}|_h, s^*_i|_h) \succsim_i|_h O_h(s^*_{-i}|_h, s_i)$$

for every strategy s_i of player i in $\Gamma(h)$

Nash equilibrium and subgame perfect equilibrium

Definition

A Nash equilibrium of $\langle N, H, P, (\succsim_i)\rangle$ is a strategy profile s^* such that for every player $i \in N$,

$$O(s^*_{-i}, s^*_i) \succsim_i O(s^*_{-i}, s_i) \text{ for every strategy } s_i \text{ of player } i$$

Definition

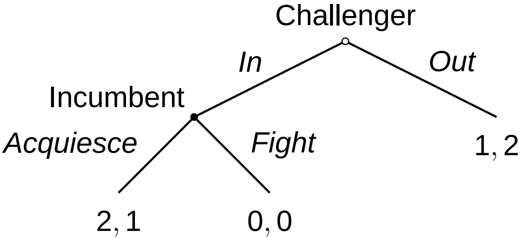
A subgame perfect equilibrium of $\langle N, H, P, (\succsim_i)\rangle$ is a strategy profile s^* such that for every player $i \in N$ and every nonterminal history $h \in H \setminus Z(H)$ for which $P(h) = i$,

$$O_h(s^*_{-i}|_h, s^*_i|_h) \succsim_i|_h O_h(s^*_{-i}|_h, s_i)$$

$$\text{for every strategy } s_i \text{ of player } i \text{ in } \Gamma(h)$$

- Every subgame perfect equilibrium is a Nash equilibrium
- Not every Nash equilibrium is a subgame perfect equilibrium

Example: entry game



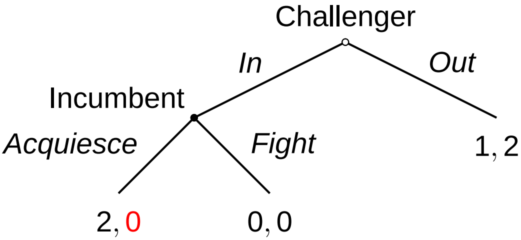
(*In*, *Acquiesce*) Subgame perfect equilibrium:

- *In* optimal at start of game, given Incumbent's strategy
- *Acquiesce* optimal in subgame following *In*

(*Out*, *Fight*) Not subgame perfect equilibrium:

- *Out* optimal at start of game
- But *Fight* not optimal in subgame following *In*

Example: variant of entry game



$(In, Acquiesce)$ Subgame perfect equilibrium:

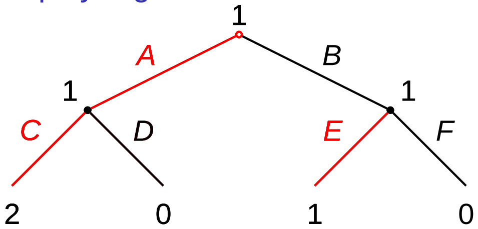
- *In* optimal at start of game, given Incumbent's strategy
- *Acquiesce* optimal in subgame following *In*

$(Out, Fight)$ Subgame perfect equilibrium:

- *Out* optimal at start of game, given Incumbent's strategy
- *Fight* optimal after history *In*

Checking strategy profile is SPE

Example: one-player game



Is strategy *ACE* optimal?

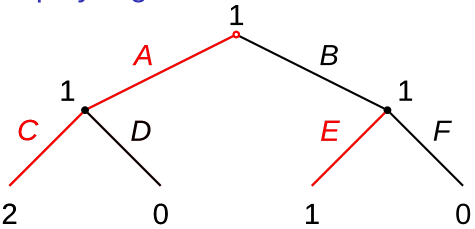
Subgame following *A* Payoff to *C* higher than payoff to *D* $\Rightarrow$ *ACE* is optimal in subgame

Subgame following *B* Payoff to *E* higher than payoff to *F* $\Rightarrow$ *ACE* is optimal in subgame

Whole game Payoff to *ACE* at least as high as payoffs to *ACF*, *ADE*, *ADF*, *BCE*, *BCF*, *BDE*, and *BDF* $\Rightarrow$ *ACE* is optimal in whole game

Checking strategy profile is SPE

Example: one-player game



Is strategy *ACE* optimal?

First two steps ⇒

- if player initially chooses *A* then *C* is optimal in subgame following *A*
- if player initially chooses *B* then *E* is optimal in subgame following *B*

So when considering whole game, need to compare only the strategies *ACE* and *BCE*

One-deviation property

To check optimality of strategy, need to check only whether player can increase her payoff by changing her action at start of each subgame, *holding the rest of her strategy fixed*

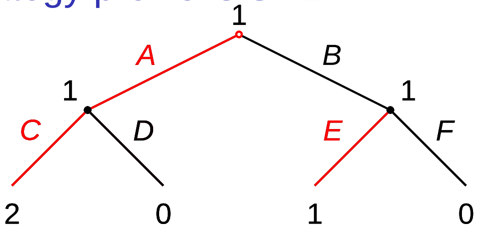
A strategy profile in an extensive game with perfect information satisfies *one-deviation property* if

> no player can increase her payoff in any subgame by changing only her action at the start of the subgame, given the other players' strategies

Proposition (*One-deviation property*)

A strategy profile in a finite horizon extensive game with perfect information is a subgame perfect equilibrium if and only if it satisfies the one-deviation property

Checking strategy profile is SPE



Is strategy *ACE* optimal? Use one-deviation property:

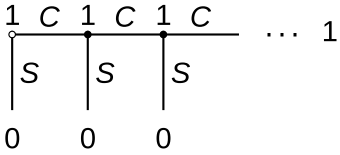
Subgame following *A* Payoff to $C >$ payoff to $D \Rightarrow$ player cannot increase payoff by changing action at start of subgame

Subgame following *B* Payoff to $E >$ payoff to $F \Rightarrow$ player cannot increase payoff by changing action at start of subgame

Whole game $A \Rightarrow$ payoff 2 and $B \Rightarrow$ payoff 1, *given rest of strategy*, so player cannot increase payoff by changing her action at start of subgame

Example

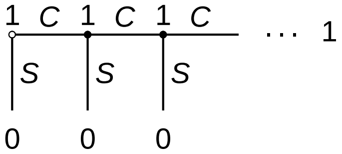
Consider following *infinite horizon* game



- One player
- Terminal histories: S, (C, S), (C, C, S), (C, C, C, S), $\ldots$, and infinite sequence $(C, C, \ldots)$
- Payoffs: 0 to every terminal history except $(C, C, \ldots)$
- Subgame perfect equilibrium: $(C, C, \ldots) \Rightarrow$ payoff 1
- Consider strategy $(S, S, \ldots)$: does it satisfy one-deviation property?
  - Yes: player cannot increase her payoff by deviating from S to C at start of any subgame, *given rest of strategy*

Example

Consider following *infinite horizon* game



- Example shows that the assumption of finite hor izon cannot be removed from result

Proposition (*One-deviation property*)

A strategy profile in a finite horizon extensive game with perfect information is a subgame perfect equilibrium if and only if it satisfies the one-deviation property

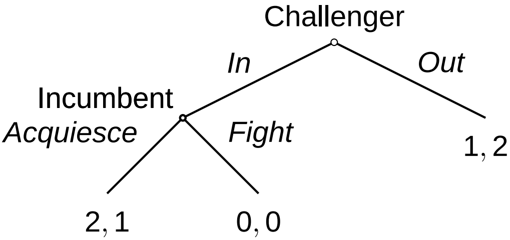
Backward induction in finite horizon games

- If strategy profile in finite horizon game satisfies one-deviation property, it is subgame perfect equilibrium
- How to *find* subgame perfect equilibria?

Backward induction

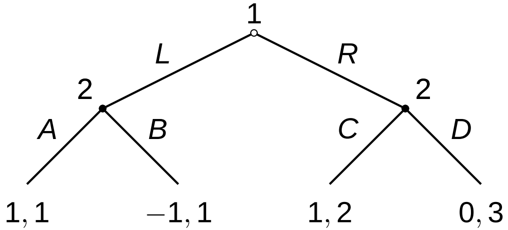
- Start by finding optimal action in every subgame of length one (at “end” of game)
- Given optimal actions in subgames of length one, find optimal action in each subgame of length two
- Continue to work backwards to start of game
- In finite horizon game, strategy profile constructed satisfies one-deviation property and hence is subgame perfect equilibrium

Example: entry game



- One subgame of length 1, following history *In*: optimal action (of Incumbent) is *Acquiesce*
- One subgame of length 2 (whole game): optimal action (of Challenger), given outcome in subgame of length 1, is *In*
- Thus game has unique subgame perfect equilibrium, $(In, Acquiesce)$

Example: game with indifference between outcomes



- Subgames of length one:
  - following L: A and B are both optimal
  - following R: D is optimal
- Subgame of length two (whole game): Need to consider separately each collection of optimal actions in subgames of length one:
  - AD: L is optimal
  - BD: R is optimal
- Thus *two* subgame perfect equilibria:
  - (L, AD)
  - (R, BD)

Backward induction in finite horizon games

Backward induction procedure in finite horizon game constructs set of strategy profiles satisfying one-deviation property

Proposition

The set of strategy profiles found by the procedure of backward induction in a finite horizon game is the set of subgame perfect equilibria.

In *finite* game, at least one action at start of every subgame is optimal, given any collection of following actions, so such a game has a subgame perfect equilibrium

Proposition

Every finite extensive game with perfect information has a subgame perfect equilibrium.