ECO2030: Microeconomic Theory II, module 1

Lecture 5

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Campbell's
CONDENSED
VEGETABLE
SOUP
MADE WITH BEEF STOCK

Auctions

- Auction: mechanism in which traders submit bids and winner and price depend on bids
- Goods sold by auction:
  - Art (e.g. Sotheby's, founded 1744; Christie's, founded 1766)
  - Fish, cattle, flowers
  - Treasury bills
  - Ads on search engines (Google ad revenue second quarter 2016 about $19 billion)
  - Oil tracts, timber
  - Wireless spectrum (for cell phones, TV, ...): revenue from 2008 auction in Canada $4.25 billion
  - Government contracts
  - Everything: eBay's 2015 sales revenue $22 billion
  - Repairs to your house

Auctions

Many versions:

- Bids submitted sequentially (Christie's, Sotheby's, eBay) or simultaneously (sealed-bid)
- Sale price $=$ highest bid or some other price
- Single object for sale (e.g. work of art), or many interrelated objects (e.g. licences to use radio spectrum for wireless communication in connected areas)
- Each player's value of object may be independent of other players' valuations, or dependent on them

Single object independent private values sealed-bid auction

- Single object for sale
- n bidders
- Each bidder's valuation of object known to her, fixed independently of other bidders' valuations
- Each bidder doesn't know *other* bidders' valuations; believes each is drawn independently from same distribution F on $[\underline{v}, \overline{v}]$
- Bids submitted simultaneously
- Bidder who submits highest bid wins

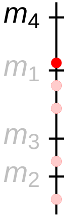
Second-price auction

- Price paid by winner is highest losing bid (absent ties, second highest bid)
- One reason why rule is interesting: models open oral ascending (“English”) auction:
  - given independent private values, bidders don’t learn from others’ bids
  - so can model players’ strategies as limit bids (price at which to drop out)
  - price stops increasing when $n - 1$ bidders have dropped out $\Rightarrow$ price paid by winner is slightly above second highest limit bid

Second-price auction

Ascending auction

- Suppose 4 bidders with limit bids m_1, m_2, m_3, and m_4
- Price starts low: everyone wants to bid
- As price rises, bidders drop out
- Once price goes above m_1, bidding stops $\Rightarrow$ bidder 4 wins and pays price slightly above m_1—*second highest* limit bid



Second-price auction

Bayesian game

Players $N = \{1, \dots, n\}$ (bidders)

States $\Omega = \{(v_1, \dots, v_n) : \underline{v} \leq v_i \leq \overline{v} \text{ for all } i\}$

Actions $A_i = \mathbb{R}_+$ for each $i \in N$ (bid $=$ any nonnegative number)

Signals $T_i = [\underline{v}, \overline{v}]$ and $\tau_i(v_1, \dots, v_n) = v_i$ for all $(v_1, \dots, v_n)$ and all $i \in N$ (each player knows own valuation)

Beliefs Every player believes that the other players' valuations are independent draws from F: each player i assigns probability $\Pi_{j=1}^{n} F(v_j)$ to the set of states in which the valuation of every player j is at most v_j

Second-price auction

Bayesian game continued

Payoff functions

$$u_i((b_1,\ldots,b_n),(v_1,\ldots,v_n)) = \begin{cases} v_i - \max_{j\neq i} b_j & \text{if } b_j < b_i \text{ for all } j \neq i \\ (v_i - b_i)/m & \text{if } b_j \leq b_i \text{ for all } j \in N \text{ and} \\ & \quad |\{j \in N : b_j = b_i\}| = m \geq 2 \\ 0 & \text{if } b_j > b_i \text{ for some } j \neq i \end{cases}$$

Single object independent private value sealed-bid auction: Second-price rule

Bayesian game continued

Payoff functions

$$u_i((b_1, \dots, b_n), (v_1, \dots, v_n)) = \begin{cases} \dfrac{v_i - \max_{j \neq i} b_j}{|\{j \in N : b_j = b_i\}|} & \text{if } b_j \leq b_i \text{ for all } j \in N \\ 0 & \text{if } b_j > b_i \text{ for some } j \neq i. \end{cases}$$

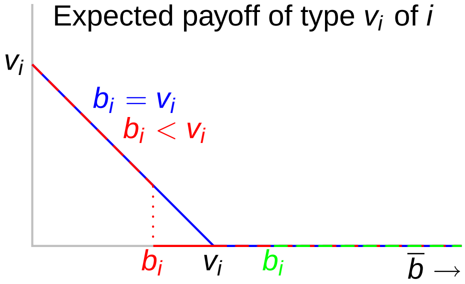
Notes

- bidders risk-neutral
- auction symmetric (all valuations drawn from same distribution)

Second-price auction

Proposition

For type v_i of player i, the bid v_i weakly dominates all other bids.

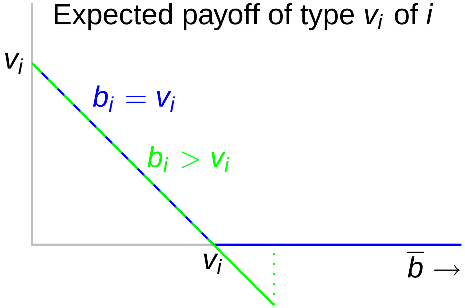


$\Rightarrow$ bid v_i weakly dominates bid $b_i < v_i$

Second-price auction

Proposition

For type v_i of player i, the bid v_i weakly dominates all other bids.



$\Rightarrow$ bid v_i weakly dominates bid $b_i > v_i$

Second-price auction

Because a player's bidding her valuation weakly dominates all her other actions . . .

Proposition

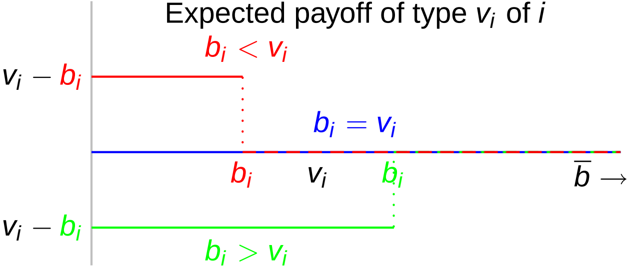
An independent private values second-price sealed-bid auction has a Nash equilibrium in which every type of every player bids her valuation.

The game has also *other* equilibria, but we select this one as "distinguished"

First-price auction

A player's bid equal to her valuation does *not* weakly dominate all other bids in a first-price auction:

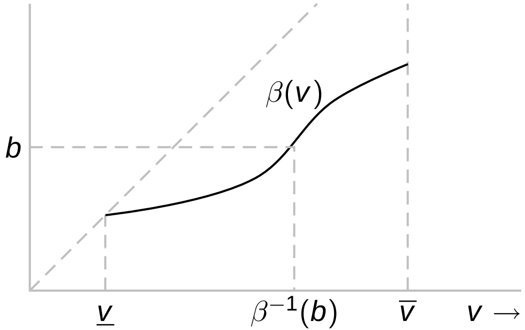
- v_i weakly dominates higher bids
- but not lower bids
- In fact, any bid $b_i < v_i$ weakly dominates $b_i = v_i$



First-price auction

Nash equilibrium

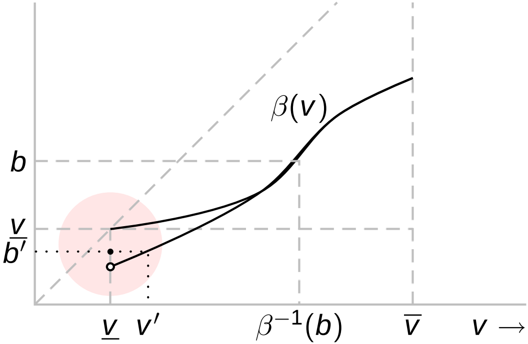
- Denote bid of type v_i of player i by $\beta_i(v_i)$
- Guess Nash equilibrium in which
  - $\beta_i(v_i) \leq v_i$ for all v_i
  - $\beta_i = \beta$ for all $i \in N$ (symmetric equilibrium)
  - β is increasing (higher valuation $\Rightarrow$ higher bid) and continuous



First-price auction

Argument that $\beta(\underline{v}) = \underline{v}$:

- $\beta(\underline{v}) < \underline{v} \Rightarrow \beta(v) < \underline{v}$ for v close to $\underline{v}$ (given β continuous)
- Player with valuation $\underline{v}$ wins with probability 0
- If player with valuation $\underline{v}$ increases bid to b' she wins when highest other valuation $< v' \Rightarrow$ with positive probability



First-price auction

- Consider player i
- Suppose that all other players bid according to β
- For equilibrium, $\beta(v)$ must be optimal for every type v of player i, given other players' bids
- That is, for all v

$$\text{bid of } \beta(v) \text{ maximizes expected payoff of type } v$$

$\Rightarrow$

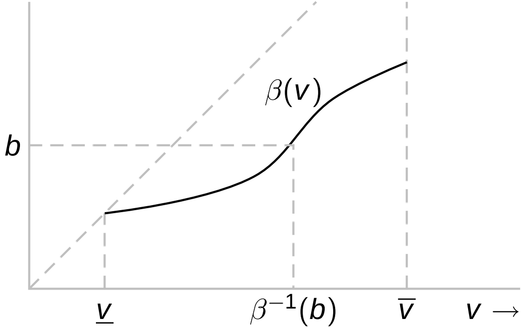
$$\beta(v) \text{ solves } \max_b (v - b) \Pr(\text{all other bids} < b)$$

First-price auction

$$\beta(v) \text{ solves } \max_b (v - b)\Pr(\text{all other bids} < b) \text{ for all } v$$

Now,

$$\Pr(\text{all other bids} < b) = \Pr(\text{all other valuations} < \beta^{-1}(b))$$
$$= \Pr(\text{highest of other valuations} < \beta^{-1}(b))$$



First-price auction

$$\beta(v) \text{ solves } \max_b (v-b)\Pr(\text{all other bids} < b) \text{ for all } v$$

Now,

$$\Pr(\text{all other bids} < b) = \Pr(\text{all other valuations} < \beta^{-1}(b))$$
$$= \Pr(\text{highest of other valuations} < \beta^{-1}(b))$$

Let

$$\mathbf{X} = \text{highest of } n-1 \text{ randomly selected valuations}$$
$$H = \text{cumulative distribution function of } \mathbf{X}$$
$$\Rightarrow \Pr(\text{highest of other valuations} < \beta^{-1}(b)) = H(\beta^{-1}(b))$$

So equilibrium condition is

$$\beta(v) \text{ solves } \max_b (v-b)H(\beta^{-1}(b)) \text{ for all } v$$

First-price auction

$$\beta(v) \text{ solves } \max_b (v-b)H(\beta^{-1}(b)) \text{ for all } v$$

If H is differentiable then

$$b^* \text{ solves } \max_b (v-b)H(\beta^{-1}(b))$$

$\Rightarrow$ at least for $b^* > 0$

$$-H(\beta^{-1}(b^*)) + (v-b^*)H'(\beta^{-1}(b^*))(\beta^{-1})'(b^*) = 0$$

Thus

$$\beta(v) \text{ solves } \max_b (v-b)H(\beta^{-1}(b)) \text{ for all } v$$

$\Rightarrow$

$$-H(\beta^{-1}(\beta(v))) + (v-\beta(v))H'(\beta^{-1}(\beta(v)))(\beta^{-1})'(\beta(v)) = 0 \text{ for all } v$$

First-price auction

$$\beta(v) \text{ solves } \max_b (v-b)H(\beta^{-1}(b)) \text{ for all } v$$

$\Rightarrow$

$$-H(\beta^{-1}(\beta(v))) + (v-\beta(v))H'(\beta^{-1}(\beta(v)))(\beta^{-1})'(\beta(v)) = 0 \text{ for all } v$$

$\Rightarrow$

$$-H(v) + (v-\beta(v))H'(v)\frac{1}{\beta'(v)} = 0 \text{ for all } v$$

Recall: for differentiable function f with differentiable inverse,

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

(differentiate identity $f(f^{-1}(x)) = x$)

First-price auction

So for equilibrium,

$$-H(v) + (v - \beta(v))H'(v)\frac{1}{\beta'(v)} = 0 \quad \text{for all } v$$

$\Rightarrow$

$$\beta'(v)H(v) + \beta(v)H'(v) = vH'(v) \quad \text{for all } v$$

Integrate both sides:

$$\beta(v)H(v) = \int_{\underline{v}}^{v} xH'(x)\,dx + C$$

Now, $H(\underline{v}) = 0$ and β is bounded $\Rightarrow C = 0$, so

$$\beta(v) = \frac{\int_{\underline{v}}^{v} xH'(x)\,dx}{H(v)} \text{ for all } v \in [\underline{v}, \overline{v}]$$

First-price auction

$$\beta^*(v) = \frac{\int_{\underline{v}}^{v} xH'(x)\,dx}{H(v)} \text{ for all } v \in [\underline{v}, \overline{v}].$$

Recall:

$$H = \text{cumulative distribution function of } \mathbf{X}$$
$$\mathbf{X} = \text{highest of } n-1 \text{ randomly selected valuations}$$

So:

$$\beta^*(v) = \frac{\int_{\underline{v}}^{v} xH'(x)\,dx}{H(v)} = \mathrm{E}(\mathbf{X} \mid \mathbf{X} < v) \text{ for all } v \in [\underline{v}, \overline{v}]$$

$\Rightarrow \beta^*$ is increasing $\Rightarrow$ strategy profile in which each type v of each player i bids $\beta^*(v)$ is Nash equilibrium of first-price auction

First-price auction

Proposition

An independent private values first-price sealed-bid auction has a Nash equilibrium in which the bid of each type v of each player is

$$\mathrm{E}(\mathbf{X} \mid \mathbf{X} < v)$$

First-price auction

Interpretation

$$\beta^*(v) = \mathrm{E}(\mathbf{X} \mid \mathbf{X} < v) \text{ for all } v \in [\underline{v}, \overline{v}]$$

Player with valuation v bids expected value of highest of other players' valuations over all lists of other players' valuations in which highest valuation is less than v

Each bidder asks: Over all cases in which my valuation is the highest, what is expected value of highest of other players' valuations? She bids this expected value

Alternatively: player with valuation v bids expected value of highest of the other players' valuations conditional on her winning

First-price auction

$$\beta^*(v) = \mathrm{E}(\mathbf{X} \mid \mathbf{X} < v) \quad \text{for all } v \in [\underline{v}, \overline{v}]$$

Other equilibria exist, but we select this equilibrium as the "distinguished" equilibrium

Comparative static: $n \uparrow \Rightarrow \beta(v) \uparrow$ for all v
(because expected value of highest of other players' valuations increases)

When n is very large, $\mathrm{E}(\mathbf{X} \mid \mathbf{X} < v)$ is close to v

Comparison of first- and second-price auctions

First-price auction

- Bidder with valuation v bids $E(\mathbf{X} \mid \mathbf{X} < v)$
- Winner is bidder with highest valuation v, who pays $E(\mathbf{X} \mid \mathbf{X} < v)$

Second-price auction

- Bidder with valuation v bids v
- Winner is bidder with highest valuation v, who pays price equal to second-highest bid, the expected value of which is $E(\mathbf{X} \mid \mathbf{X} < v)$

Proposition (*Revenue equivalence*)

If each bidder is risk neutral then in a symmetric independent private values sealed-bid auction the distinguished Nash equilibria under first- and second-price rules yield the same expected revenue

Common value auctions

- In many auctions, bidders' valuations are not independent
- Instead, bidders' valuations may be related to each other
- Even a buyer of a work of art may care about its resale value, which depends on other people's valuations of it
- Interdependence of values introduces considerations not present when values are independent

Common value auctions

Drilling for oil

- All firms value oil in the same way
- But no firm knows amount available
- Each firm i privately takes a sample, which generates a signal s_i about amount available
- Samples differ, so firms' estimates of amount available differ
- *If* firm i were to know all firms' signals, $(s_1, \ldots, s_n)$, then its estimate of the amount available would be $v_i(s_1, \ldots, s_n)$
- Assume v_i is increasing in s_i and nondecreasing in s_j for $j \neq i$
- Special case: $v_i(s_1, \ldots, s_n) = s_i$ (private valuations)
- Special case: $v_i = u$, same for all i (pure common values)

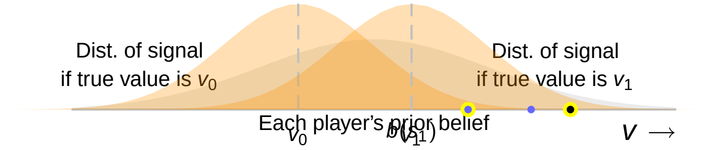
Common value auctions

Drilling for oil: "mineral rights" model

- $v_i = u$, same for all i (pure common values)
- Value of random variable v is true value of oil
- Players' signals are independent conditional on v and the expectation of each s_i equal to v

Common value auctions

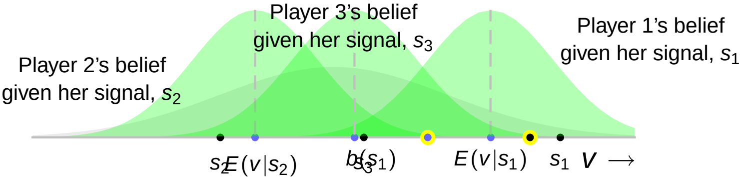
Drilling for oil



- Each player sees only her *own* signal
- Signal and prior belief $\Rightarrow$ posterior distribution of v (via Bayes' law)
- Different players get different signals, so their estimates of the value based on these signals differ

Common value auctions

Drilling for oil



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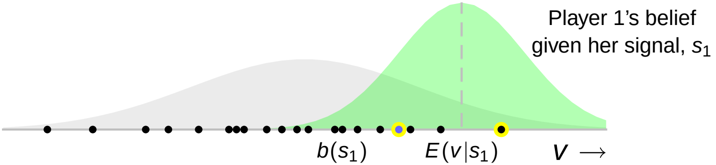
Common value auctions

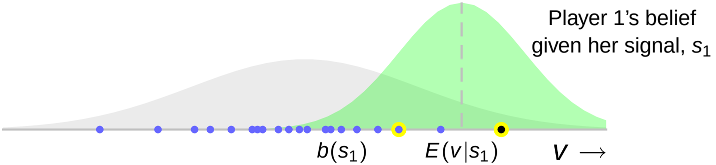
Drilling for oil



- Each black dot represents the signal received by a player
- Each blue dot represents the expectation of v given the corresponding signal—that is, $E(v \mid \text{signal is } s_i)$

Common value auctions

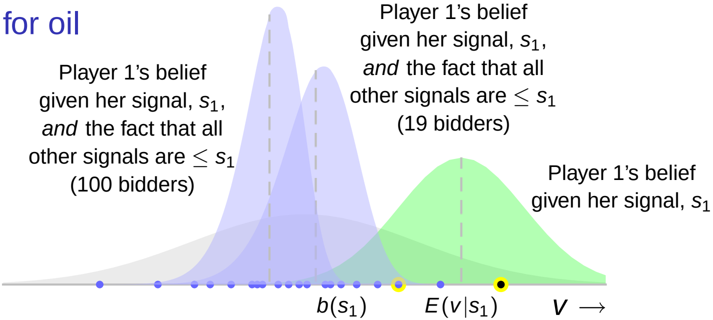
Drilling for oil



- Consider second-price auction
- Suppose that each player's bid is the expectation of the value based solely on her own signal
- Then player with highest signal wins and pays price equal to expected value of v given second-highest signal

Common value auctions

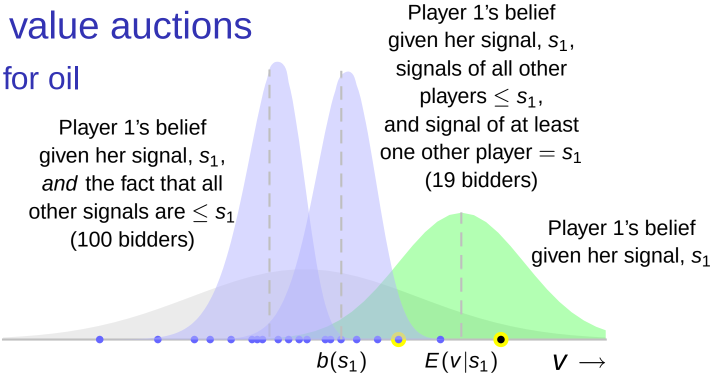
Drilling for oil



- The fact that she wins tells her that all other signals are less than hers
- Given this information, she believes that v is likely to be less than her estimate based solely on her own signal
- Typically, probability that second highest bid will exceed actual value is high, especially with many bidders
- Effect is known as winner's curse

Common value auctions

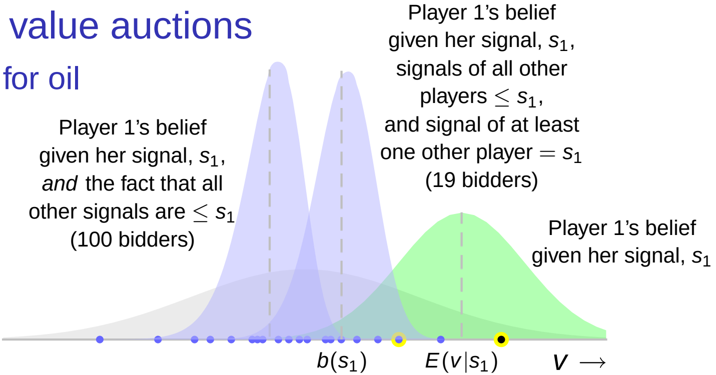
Drilling for oil



- When formulating bid, player should take into account that if she wins, all other players' signals will be lower than hers
- She should take this information into account, and base her bid on estimate of value *conditional on her winning* (given other players' strategies)

Common value auctions

Drilling for oil



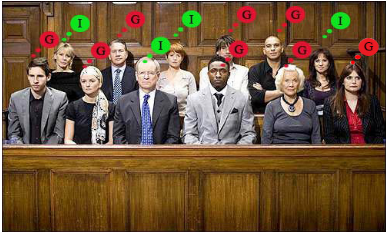
- In Nash equilibrium of second-price auction, player i with signal s_i bids

$$b(s_i) = E(v \mid i\text{'s signal is } s_i, \text{ signals of all other players are } \leq s_i, \text{ and signal of at least one other player is equal to } s_i)$$

- This expectation is typically much less than $E(v \mid s_i)$

Juries

- n jurors
- Each juror has same prior belief that defendant is guilty with probability π
- All jurors share same goal: convict guilty person, acquit innocent one
- But jurors may interpret evidence differently



Juries

Information structure

- Model each juror as receiving a *signal* from the evidence
- If defendant guilty, more likely to get *guilty* signal; if defendant innocent, more likely to get *innocent* signal

$p > \frac{1}{2}$ $\qquad$ $q > \frac{1}{2}$

- Jurors do not share signals; they do not deliberate

Juries

Actions and outcome

- After all jurors have received their signals, each juror votes to *acquit* or *convict*
- Defendant is convicted only if *all* jurors vote to *convict*

Juries

Bayesian game

Players The n jurors

States $\{(X, s_1, \ldots, s_n) : X \in \{G, I\}$ and $s_i \in \{g, b\}$ for $i = 1, \ldots, n\}$

Actions $A_i = \{Convict, Acquit\}$ for $i = 1, \ldots, n$

Signals $T_i = \{g, b\}$ and $\tau_i(X, s_1, \ldots, s_n) = s_i$ for $i = 1, \ldots, n$

Beliefs For state $(G, s_1, \ldots, s_n)$ in which k signals are g and $n - k$ are b, common prior probability is $\pi p^k (1-p)^{n-k}$; for state $(I, s_1, \ldots, s_n)$ in which k signals are g and $n - k$ are b, common prior probability is $(1-\pi)(1-q)^k q^{n-k}$

Juries

Bayesian game, continued

Payoffs

$$u_i(a,\omega) = \begin{cases} 0 & \text{if } \omega_1 = G \text{ and } a_j = \textit{Convict} \text{ for all } j \\ 0 & \text{if } \omega_1 = I \text{ and } a_j = \textit{Acquit} \text{ for some } j \\ -z & \text{if } \omega_1 = I \text{ and } a_j = \textit{Convict} \text{ for all } j \\ -(1-z) & \text{if } \omega_1 = G \text{ and } a_j = \textit{Acquit} \text{ for some } j \end{cases}$$

with $0 < z < 1$

Interpretation of payoffs

- Let posterior probability juror assigns to guilt be r
- Juror prefers acquittal if $-r(1-z) > -(1-r)z$, or $r < z$
- So z is cutoff probability for juror's preferring to convict

Juries

Is the outcome in which every juror votes according to her signal an equilibrium?

Juror's decision

- Consider juror i
- Suppose that every *other* juror votes according to her signal

other jurors' signals

		all *innocent*	$n-2$ *innocent*	...	1 *innocent*	all *guilty*
juror i	Acquit	A	A	...	A	A
	Convict	A	A	...	A	C

Outcome (A = acquittal, C = conviction)

Juries

Juror's decision

		other jurors' signals				
		all *innocent*	$n-2$ *innocent*	...	1 *innocent*	all *guilty*
juror i	Acquit	A	A	...	A	A
	Convict	A	A	...	A	C

Outcome (A = acquittal, C = conviction)

- How should juror i vote?
- Her action makes a difference to the outcome only if all the other jurors' signals are *guilty*

Juries

Juror's decision

		other jurors' signals				
		all *innocent*	$n-2$ *innocent*	...	1 *innocent*	all *guilty*
juror i	Acquit	A	A	...	A	A
	Convict	A	A	...	A	C

Outcome (A = acquittal, C = conviction)

- Suppose her signal is *innocent*
- Then *Acquit* is optimal for her if

$$
\begin{aligned}
&-\Pr(G \mid n-1 \textit{ guilty} \text{ signals and } 1 \textit{ innocent} \text{ signal})(1-z) \\
&\quad + \Pr(I \mid n-1 \textit{ guilty} \text{ signals and } 1 \textit{ innocent} \text{ signal}) \cdot 0 \\
&\geq \Pr(G \mid n-1 \textit{ guilty} \text{ signals and } 1 \textit{ innocent} \text{ signal}) \cdot 0 \\
&\quad - \Pr(I \mid n-1 \textit{ guilty} \text{ signals and } 1 \textit{ innocent} \text{ signal}) z
\end{aligned}
$$

Juries

Juror's decision

		other jurors' signals				
		all *innocent*	$n-2$ *innocent*	...	1 *innocent*	all *guilty*
juror i	Acquit	A	A	...	A	A
	Convict	A	A	...	A	C

Outcome (A = acquittal, C = conviction)

- or

$$-\Pr(G \mid n-1 \textit{ guilty} \text{ signals and 1 } \textit{innocent} \text{ signal})(1-z)$$
$$\geq -\Pr(I \mid n-1 \textit{ guilty} \text{ signals and 1 } \textit{innocent} \text{ signal})z$$

$$\Leftrightarrow$$

$$\Pr(G \mid n-1 \textit{ guilty} \text{ signals and 1 } \textit{innocent} \text{ signal})(1-z)$$
$$\leq (1-\Pr(G \mid n-1 \textit{ guilty} \text{ signals and 1 } \textit{innocent} \text{ signal}))z$$

Juries

Juror's decision

		other jurors' signals				
		all *innocent*	$n-2$ *innocent*	...	1 *innocent*	all *guilty*
juror i	Acquit	A	A	...	A	A
	Convict	A	A	...	A	C

Outcome (A = acquittal, C = conviction)

- or

$$\Pr(G \mid n-1 \textit{ guilty} \text{ signals and } 1 \textit{ innocent} \text{ signal}) \leq z$$

$$\Leftrightarrow \quad \frac{(1-p)p^{n-1}\pi}{(1-p)p^{n-1}\pi + q(1-q)^{n-1}(1-\pi)} \leq z$$

$$\Leftrightarrow \quad \frac{1}{1 + \frac{q}{1-p}\left(\frac{1-q}{p}\right)^{n-1}\frac{1-\pi}{\pi}} \leq z$$

Juries

Juror's decision

		other jurors' signals				
		all *innocent*	$n-2$ *innocent*	...	1 *innocent*	all *guilty*
juror i	Acquit	A	A	...	A	A
	Convict	A	A	...	A	C

Outcome (A = acquittal, C = conviction)

- Conclusion: given $z < 1$, for n large enough, juror with *innocent* signal optimally votes *Convict*
- Thus for n large enough, every juror's voting according to her signal is *not* a Nash equilibrium
- n may not have to be very large: if $p = q = 0.8$, $\pi = 0.5$, and $n = 12$, LHS of inequality exceeds 0.999999
- If juror with *innocent* signal optimally votes *Convict*, then so does juror with *guilty* signal

Juries

Conclusion

- If all other jurors vote according to their signals, the remaining juror should vote for *conviction* regardless of her signal
- So there is no equilibrium in which all jurors vote according to their signals
- Note that we have not determined what *is* an equilibrium