ECO2030: Microeconomic Theory II, module 1

Lecture 5

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  - Repairs to your house

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Many versions:

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- Sale price $=$ highest bid or some other price
- Single object for sale (e.g. work of art), or many interrelated objects (e.g. licences to use radio spectrum for wireless communication in connected areas)
- Each player's value of object may be independent of other players' valuations, or dependent on them

Single object independent private values sealed-bid auction

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- Bidder who submits highest bid wins

Second-price auction

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  - given independent private values, bidders don’t learn from others’ bids
  - so can model players’ strategies as limit bids (price at which to drop out)
  - price stops increasing when $n - 1$ bidders have dropped out $\Rightarrow$ price paid by winner is slightly above second highest limit bid

Second-price auction

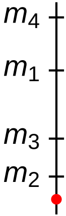
Ascending auction

- Suppose 4 bidders with limit bids m_1, m_2, m_3, and m_4

Second-price auction

Ascending auction

- Suppose 4 bidders with limit bids m_1, m_2, m_3, and m_4
- Price starts low: everyone wants to bid



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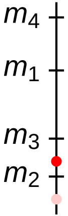

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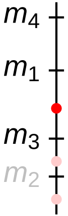

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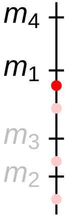

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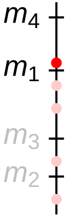

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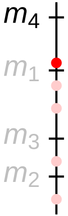
Ascending auction

- Suppose 4 bidders with limit bids m_1, m_2, m_3, and m_4
- Price starts low: everyone wants to bid
- As price rises, bidders drop out
- Once price goes above m_1, bidding stops $\Rightarrow$ bidder 4 wins and pays price slightly above m_1—*second highest* limit bid



Second-price auction

Bayesian game

Players $N = \{1, \ldots, n\}$ (bidders)

States

Actions

Signals

Beliefs

Second-price auction

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Signals $T_i = [\underline{v}, \overline{v}]$ and $\tau_i(v_1, \dots, v_n) = v_i$ for all $(v_1, \dots, v_n)$ and all $i \in N$ (each player knows own valuation)

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Beliefs Every player believes that the other players' valuations are independent draws from F: each player i assigns probability $\Pi_{j=1}^{n} F(v_j)$ to the set of states in which the valuation of every player j is at most v_j

Second-price auction

Bayesian game continued

Payoff functions

$$u_i((b_1, \ldots, b_n), (v_1, \ldots, v_n)) = \begin{cases} \\ \\ \end{cases}$$

Second-price auction

Bayesian game continued

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Bayesian game continued

Payoff functions

$$u_i((b_1,\ldots,b_n),(v_1,\ldots,v_n)) = \begin{cases} v_i - \max_{j\neq i} b_j & \text{if } b_j < b_i \text{ for all } j \neq i \\ (v_i - b_i)/m & \text{if } b_j \leq b_i \text{ for all } j \in N \text{ and} \\ & \quad |\{j \in N : b_j = b_i\}| = m \geq 2 \\ 0 & \text{if } b_j > b_i \text{ for some } j \neq i \end{cases}$$

Single object independent private value sealed-bid auction: Second-price rule

Bayesian game continued

Payoff functions

$$u_i((b_1,\ldots,b_n),(v_1,\ldots,v_n)) = \begin{cases} \dfrac{v_i - \max_{j\neq i} b_j}{|\{j \in N : b_j = b_i\}|} & \text{if } b_j \leq b_i \text{ for all } j \in N \\ 0 & \text{if } b_j > b_i \text{ for some } j \neq i. \end{cases}$$

Notes

- bidders risk-neutral
- auction symmetric (all valuations drawn from same distribution)

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Proposition

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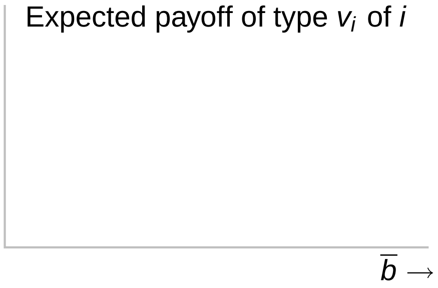

highest of other players' bids (= price)

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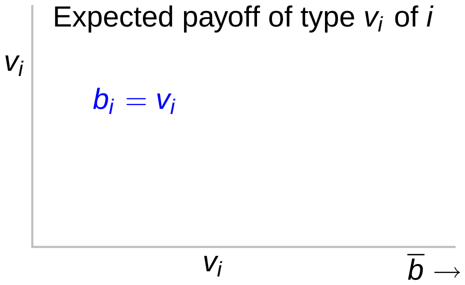

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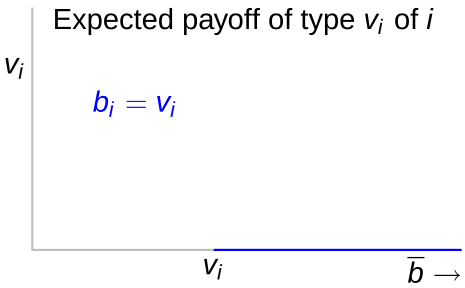

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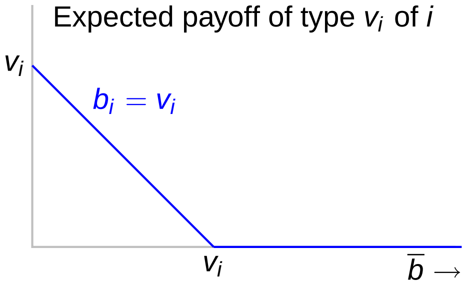

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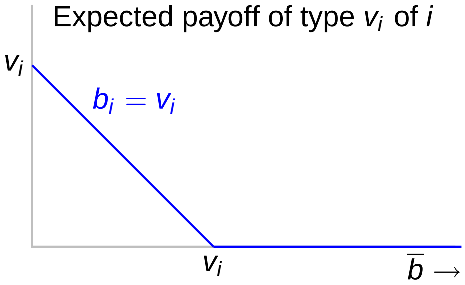

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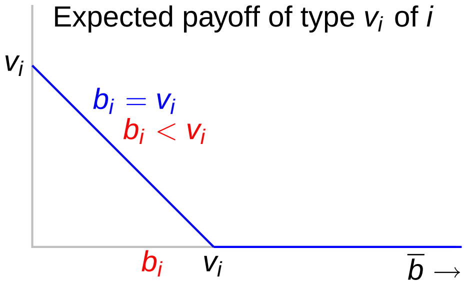

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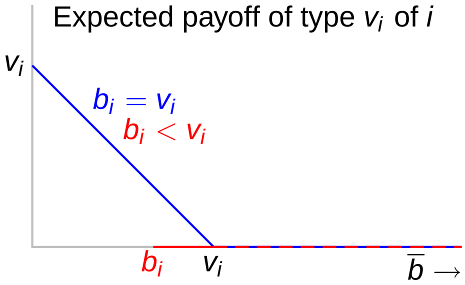

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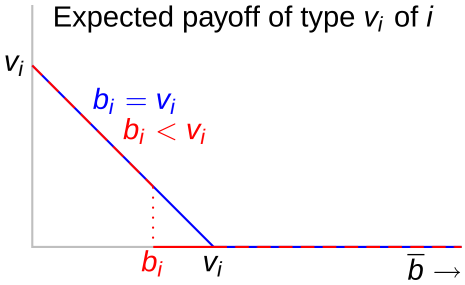

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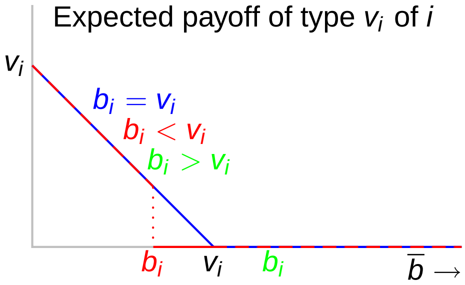

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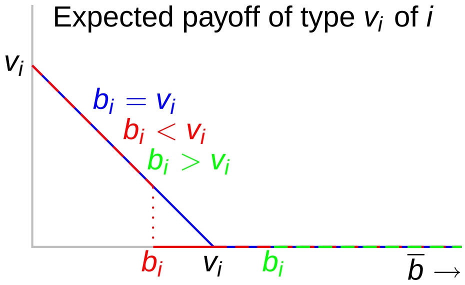

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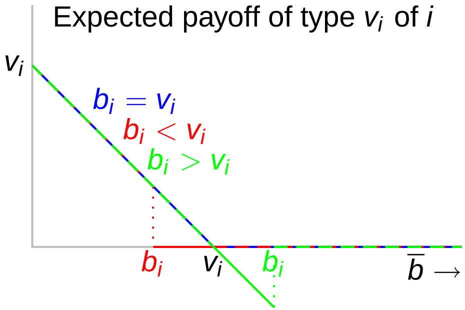

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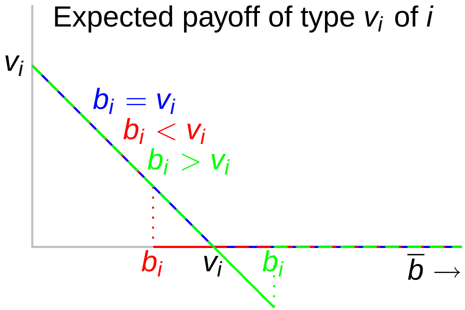

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Second-price auction

Because a player's bidding her valuation weakly dominates all her other actions . . .

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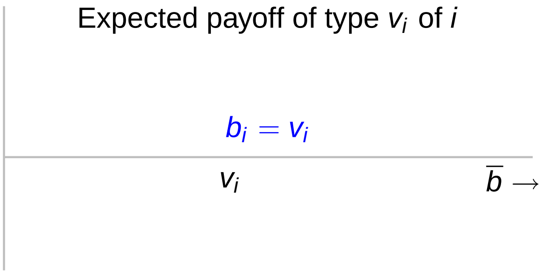
Proposition

An independent private values second-price sealed-bid auction has a Nash equilibrium in which every type of every player bids her valuation.

The game has also *other* equilibria, but we select this one as "distinguished"

First-price auction

A player's bid equal to her valuation does *not* weakly dominate all other bids in a first-price auction:



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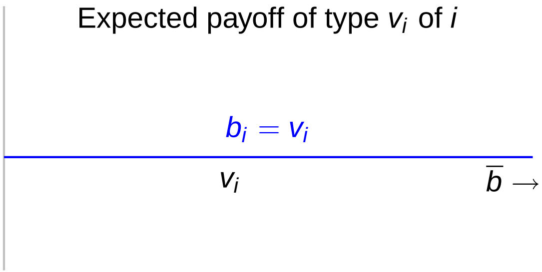

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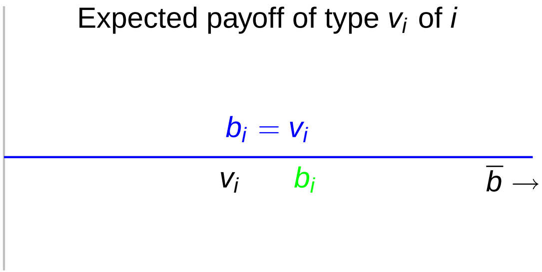

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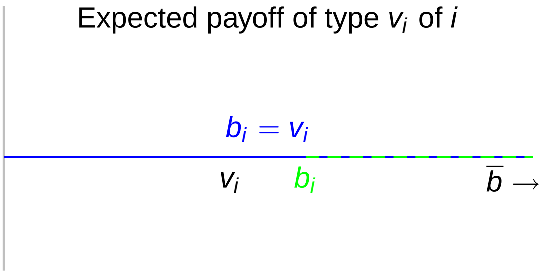

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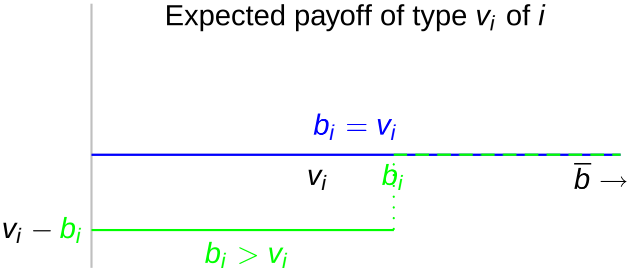
- v_i weakly dominates higher bids



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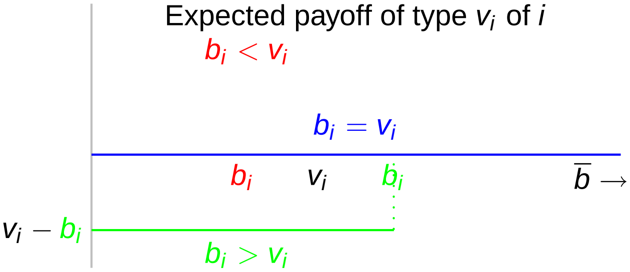

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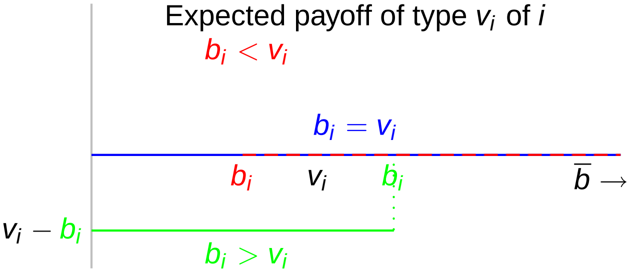

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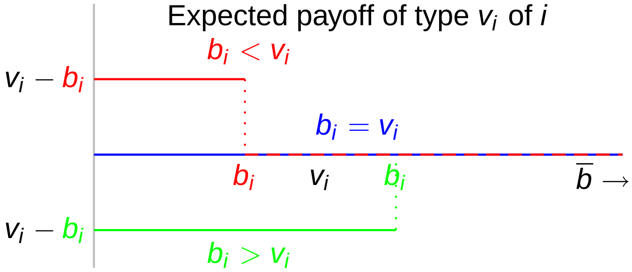

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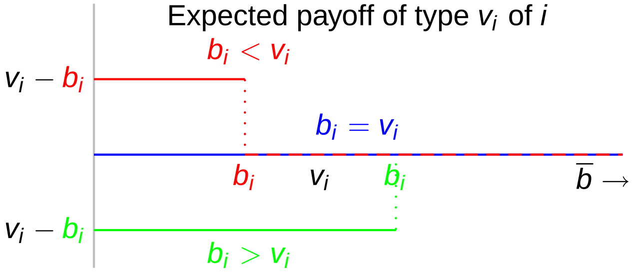
- In fact, any bid $b_i < v_i$ weakly dominates $b_i = v_i$



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Nash equilibrium

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  - $\beta_i(v_i) \leq v_i$ for all v_i

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  - $\beta_i = \beta$ for all $i \in N$ (symmetric equilibrium)

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Nash equilibrium

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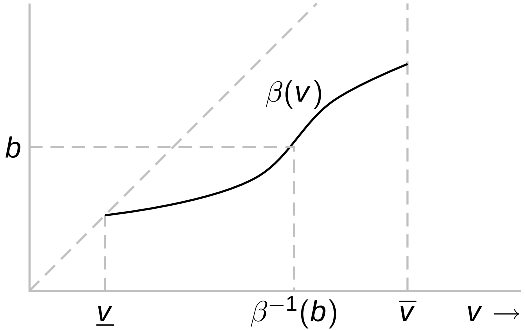
- Guess Nash equilibrium in which
  - $\beta_i(v_i) \leq v_i$ for all v_i
  - $\beta_i = \beta$ for all $i \in N$ (symmetric equilibrium)
  - β is increasing (higher valuation $\Rightarrow$ higher bid) and continuous

First-price auction

Nash equilibrium

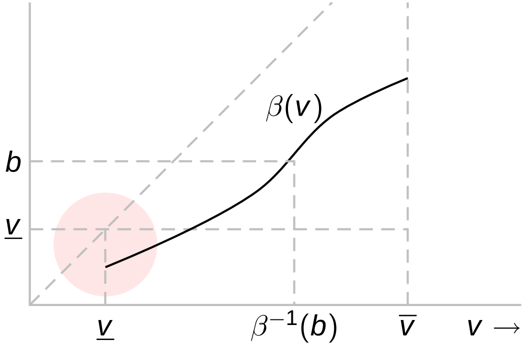
- Denote bid of type v_i of player i by $\beta_i(v_i)$
- Guess Nash equilibrium in which
  - $\beta_i(v_i) \leq v_i$ for all v_i
  - $\beta_i = \beta$ for all $i \in N$ (symmetric equilibrium)
  - β is increasing (higher valuation $\Rightarrow$ higher bid) and continuous



First-price auction

Argument that $\beta(\underline{v}) = \underline{v}$:

- $\beta(\underline{v}) < \underline{v} \Rightarrow \beta(v) < \underline{v}$ for v close to $\underline{v}$ (given β continuous)



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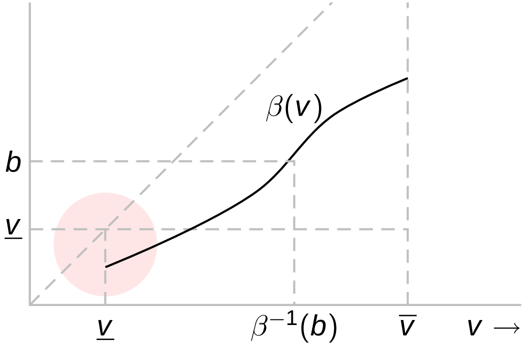

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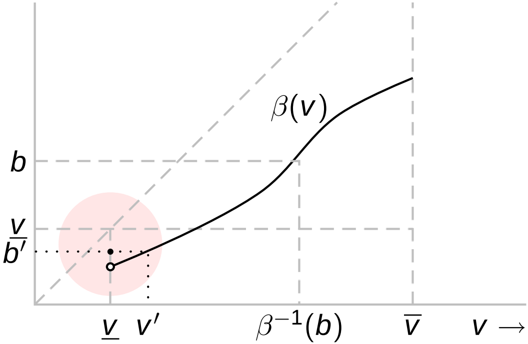

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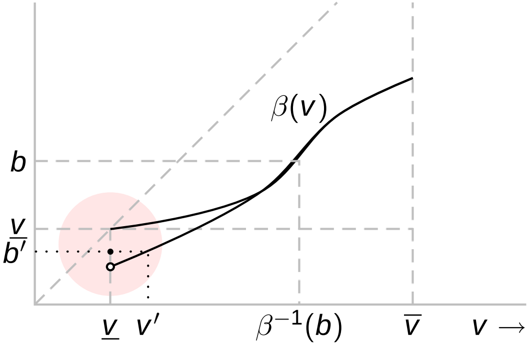

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$\Rightarrow$

$$\beta(v) \text{ solves } \max_b (v - b) \Pr(\text{all other bids} < b)$$

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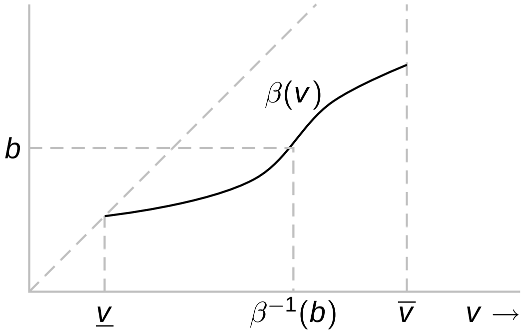

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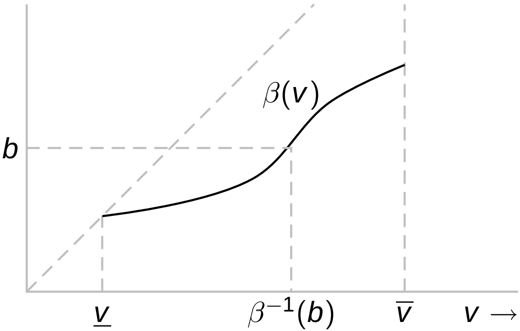

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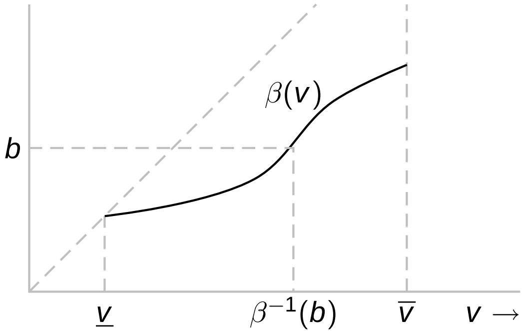

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(differentiate identity $f(f^{-1}(x)) = x$)

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Integrate both sides:

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Now, $H(\underline{v}) = 0$ and β is bounded $\Rightarrow C = 0$, so

$$\beta(v) = \frac{\int_{\underline{v}}^{v} xH'(x)\,dx}{H(v)} \text{ for all } v \in [\underline{v}, \overline{v}]$$

First-price auction

$$\beta^*(v) = \frac{\int_{\underline{v}}^{v} xH'(x)\,dx}{H(v)} \text{ for all } v \in [\underline{v}, \overline{v}].$$

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$\Rightarrow \beta^*$ is increasing $\Rightarrow$ strategy profile in which each type v of each player i bids $\beta^*(v)$ is Nash equilibrium of first-price auction

First-price auction

Proposition

An independent private values first-price sealed-bid auction has a Nash equilibrium in which the bid of each type v of each player is

$$\mathrm{E}(\mathbf{X} \mid \mathbf{X} < v)$$

First-price auction

Interpretation

$$\beta^*(v) = \mathrm{E}(\mathbf{X} \mid \mathbf{X} < v) \text{ for all } v \in [\underline{v}, \overline{v}]$$

Player with valuation v bids expected value of highest of other players' valuations over all lists of other players' valuations in which highest valuation is less than v

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Alternatively: player with valuation v bids expected value of highest of the other players' valuations conditional on her winning

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Other equilibria exist, but we select this equilibrium as the "distinguished" equilibrium

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Comparison of first- and second-price auctions

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If each bidder is risk neutral then in a symmetric independent private values sealed-bid auction the distinguished Nash equilibria under first- and second-price rules yield the same expected revenue

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Common value auctions

Drilling for oil

- All firms value oil in the same way
- But no firm knows amount available
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Common value auctions

Drilling for oil: “mineral rights” model

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Common value auctions

Drilling for oil: “mineral rights” model

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- Value of random variable v is true value of oil

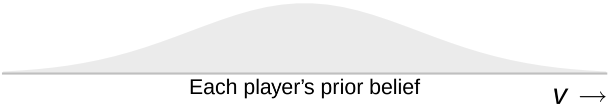
Common value auctions

Drilling for oil: "mineral rights" model

- $v_i = u$, same for all i (pure common values)
- Value of random variable v is true value of oil
- Players' signals are independent conditional on v and the expectation of each s_i equal to v

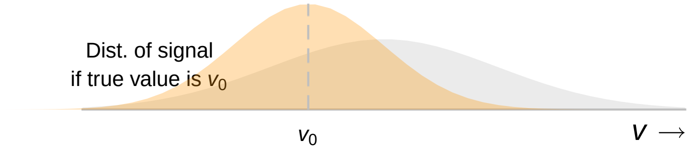
Common value auctions

Drilling for oil



Common value auctions

Drilling for oil



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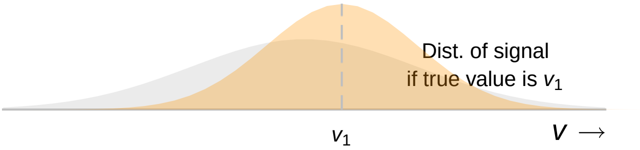


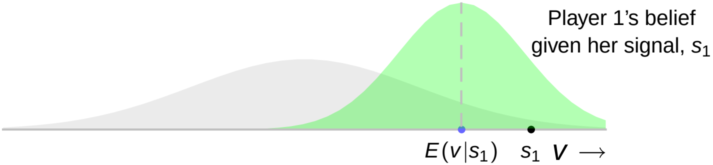
Common value auctions

Drilling for oil

- Each player sees only her *own* signal

Common value auctions

Drilling for oil



- Each player sees only her *own* signal
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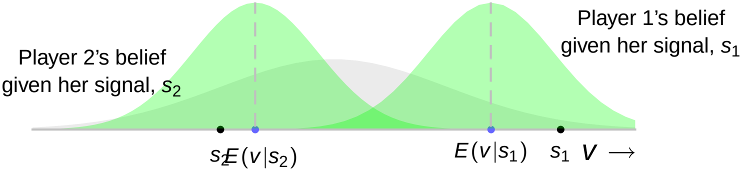
Common value auctions

Drilling for oil



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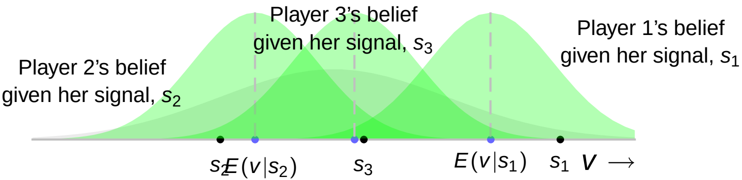

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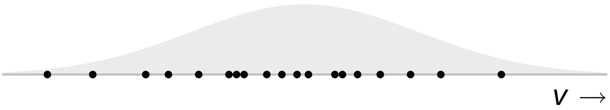

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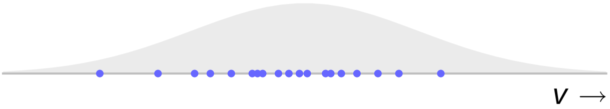
Common value auctions

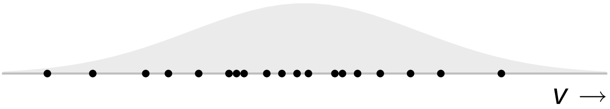
Drilling for oil



- Each black dot represents the signal received by a player
- Each blue dot represents the expectation of v given the corresponding signal—that is, $E(v \mid \text{signal is } s_i)$

Common value auctions

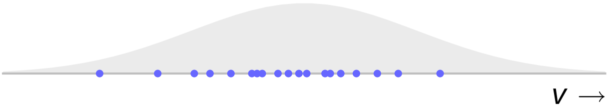
Drilling for oil



- Consider second-price auction

Common value auctions

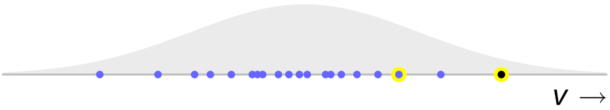
Drilling for oil



- Consider second-price auction
- Suppose that each player's bid is the expectation of the value based solely on her own signal

Common value auctions

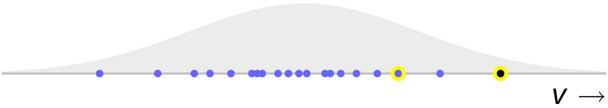
Drilling for oil



- Consider second-price auction
- Suppose that each player's bid is the expectation of the value based solely on her own signal
- Then player with highest signal wins and pays price equal to expected value of v given second-highest signal

Common value auctions

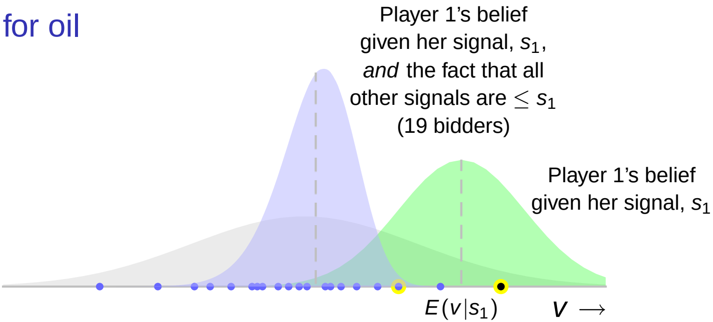
Drilling for oil



- The fact that she wins tells her that all other signals are less than hers

Common value auctions

Drilling for oil



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Common value auctions

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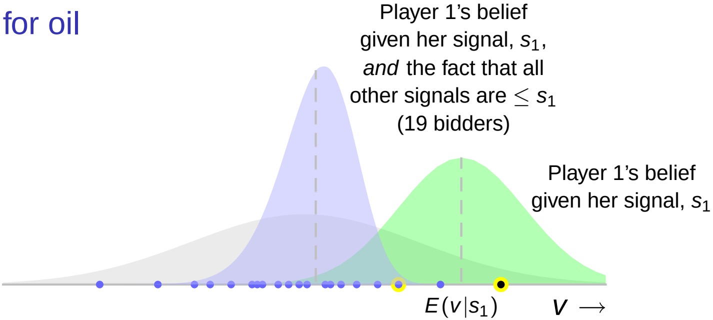

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Common value auctions

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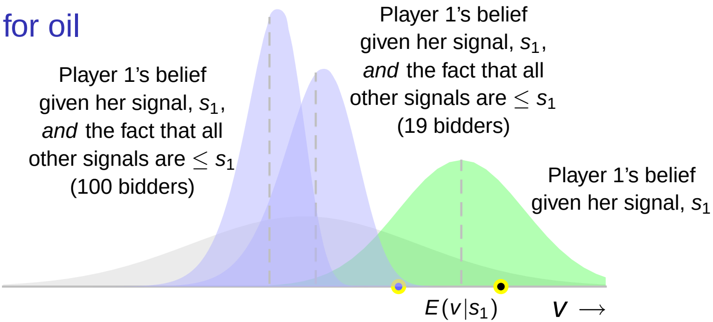

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Common value auctions

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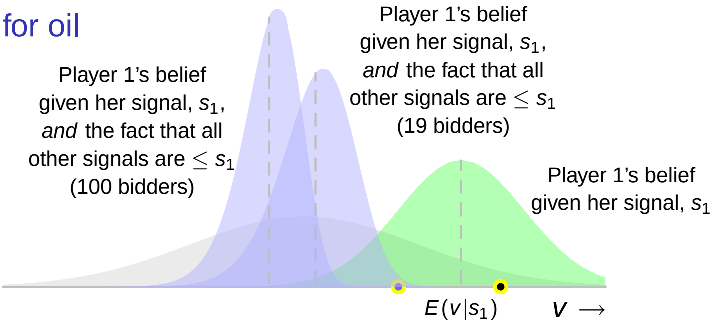

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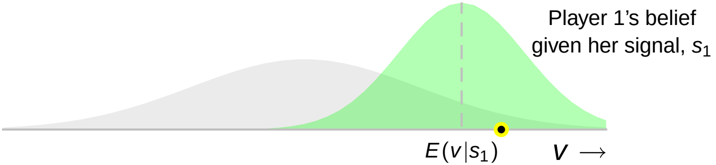
- Given this information, she believes that v is likely to be less than her estimate based solely on her own signal
- Typically, probability that second highest bid will exceed actual value is high, especially with many bidders
- Effect is known as winner's curse

Common value auctions

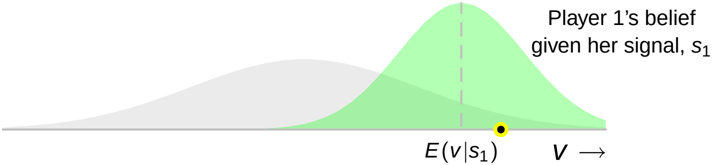
Drilling for oil



- When formulating bid, player should take into account that if she wins, all other players' signals will be lower than hers

Common value auctions

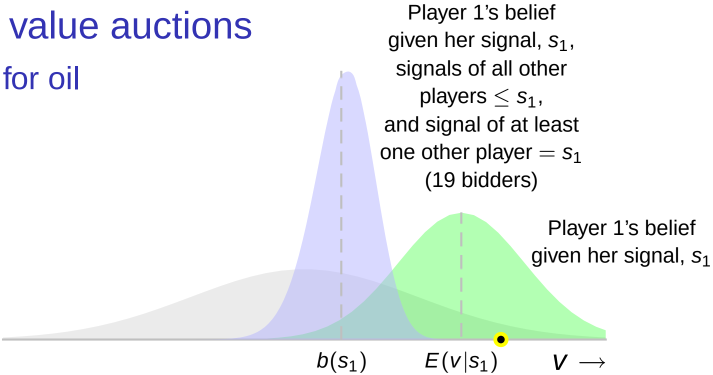
Drilling for oil



- When formulating bid, player should take into account that if she wins, all other players' signals will be lower than hers
- She should take this information into account, and base her bid on estimate of value *conditional on her winning* (given other players' strategies)

Common value auctions

Drilling for oil



- In Nash equilibrium of second-price auction, player i with signal s_i bids

$$b(s_i) = E(v \mid i\text{'s signal is } s_i\text{, signals of all other players are} \leq s_i\text{, and signal of at least one other player is equal to } s_i)$$

Common value auctions

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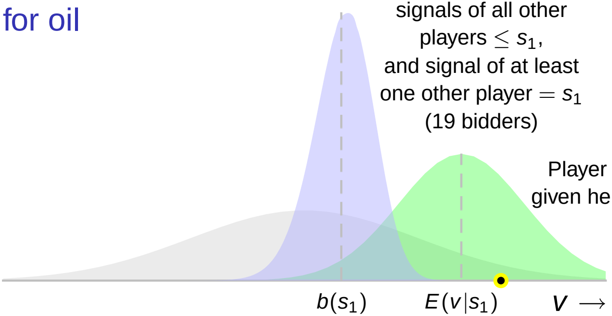

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- This expectation is typically much less than $E(v \mid s_i)$

Juries

- n jurors

Juries

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- Each juror has same prior belief that defendant is guilty with probability π

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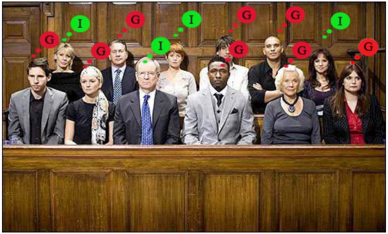

Juries

Information structure

- Model each juror as receiving a *signal* from the evidence

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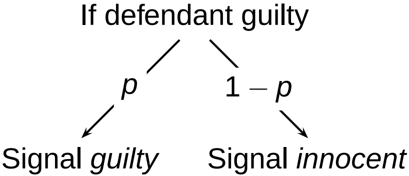

$$p > \frac{1}{2}$$

Juries

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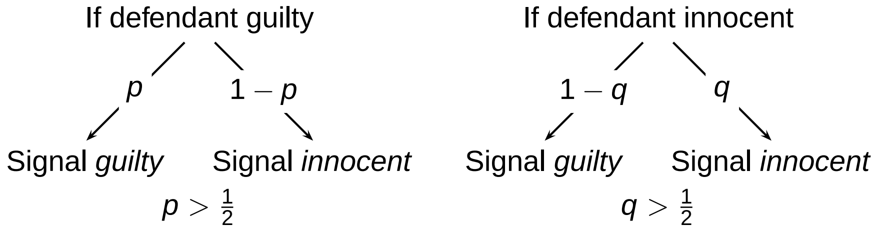

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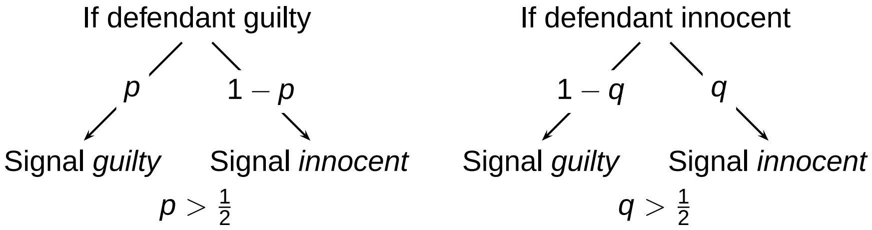

- Jurors do not share signals; they do not deliberate

Juries

Actions and outcome

- After all jurors have received their signals, each juror votes to *acquit* or *convict*

Juries

Actions and outcome

- After all jurors have received their signals, each juror votes to *acquit* or *convict*
- Defendant is convicted only if *all* jurors vote to *convict*

Juries

Bayesian game

Players The n jurors

States

Actions

Signals

Beliefs

Juries

Bayesian game

Players The n jurors

States $\{(X, s_1, \ldots, s_n) : X \in \{G, I\}$ and $s_i \in \{g, b\}$ for $i = 1, \ldots, n\}$

Actions

Signals

Beliefs

Juries

Bayesian game

$G \Rightarrow$ defendant is guilty,
$I \Rightarrow$ defendant is innocent

Players The n jurors

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Actions

Signals

Beliefs

Juries

Bayesian game

g: i's interpretation of evidence is that defendant is guilty
b: i's interpretation of evidence is that defendant is innocent

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Bayesian game

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Signals

Beliefs

Juries

Bayesian game

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Actions $A_i = \{Convict, Acquit\}$ for $i = 1, \ldots, n$

Signals $T_i = \{g, b\}$ and $\tau_i(X, s_1, \ldots, s_n) = s_i$ for $i = 1, \ldots, n$

Beliefs

Juries

Bayesian game

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Signals $T_i = \{g, b\}$ and $\tau_i(X, s_1, \ldots, s_n) = s_i$ for $i = 1, \ldots, n$

Beliefs For state $(G, s_1, \ldots, s_n)$ in which k signals are g and $n - k$ are b, common prior probability is $\pi p^k (1 - p)^{n-k}$; for state $(I, s_1, \ldots, s_n)$ in which k signals are g and $n - k$ are b, common prior probability is $(1 - \pi)(1 - q)^k q^{n-k}$

Juries

Bayesian game, continued

Payoffs

$$u_i(a, \omega) = \left\{ \right.$$

Juries

Bayesian game, continued

Payoffs

$$u_i(a, \omega) = \begin{cases} 0 & \text{if } \omega_1 = G \text{ and } a_j = \textit{Convict} \text{ for all } j \\ & \\ & \end{cases}$$

Juries

Bayesian game, continued

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with $0 < z < 1$

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Interpretation of payoffs

Juries

Bayesian game, continued

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Interpretation of payoffs

- Let posterior probability juror assigns to guilt be r

Juries

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- Let posterior probability juror assigns to guilt be r
- Juror prefers acquittal if $-r(1-z) > -(1-r)z$, or $r < z$

Juries

Bayesian game, continued

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- So z is cutoff probability for juror's preferring to convict

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Is the outcome in which every juror votes according to her signal an equilibrium?

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Juror's decision

- Consider juror i
- Suppose that every *other* juror votes according to her signal

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other jurors' signals

		all *innocent*	$n-2$ *innocent*	...	1 *innocent*	all *guilty*
juror *i*	Acquit					
	Convict					

Outcome (A = acquittal, C = conviction)

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Is the outcome in which every juror votes according to her signal an equilibrium?

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- Su… *…ther* juror votes according to her sig…

⇒ all other jurors vote to Acquit

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$\Rightarrow$ all other jurors vote to Acquit

		other jurors' signals				
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juror *i*	Acquit	*A*				
	Convict	*A*				

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other jurors’ signals

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other jurors' signals

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juror *i*	Acquit	*A*	*A*			
	Convict	*A*	*A*			

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Is the outcome in which every juror votes according to her signal an equilibrium?

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other jurors' signals

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$\Rightarrow$ all but one of the other jurors vote to Convict

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Outcome (A = acquittal, C = conviction)

Juries

Is the outcome in which every juror votes according to her signal an equilibrium?

Juror's decision

- Consider juror i
- Suppose that every *other* juror votes signal

⇒ all but one of the other jurors vote to Convict

other jurors' signals

juror i		all *innocent*	$n-2$ *innocent*	...	1 *innocent*	all *guilty*
	Acquit	A	A	...	A	
	Convict	A	A	...	A	

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Outcome (A = acquittal, C = conviction)

Juries

Juror's decision

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Outcome (A = acquittal, C = conviction)

- How should juror i vote?

Juries

Juror's decision

		other jurors' signals				
		all *innocent*	$n-2$ *innocent*	...	1 *innocent*	all *guilty*
juror i	Acquit	A	A	...	A	A
	Convict	A	A	...	A	C

Outcome (A = acquittal, C = conviction)

- How should juror i vote?
- Her action makes a difference to the outcome only if all the other jurors' signals are *guilty*

Juries

Juror's decision

		other jurors' signals				
		all *innocent*	$n-2$ *innocent*	...	1 *innocent*	all *guilty*
juror i	Acquit	A	A	...	A	A
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Outcome (A = acquittal, C = conviction)

- Suppose her signal is *innocent*

Juries

Juror's decision

		other jurors' signals				
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Outcome (A = acquittal, C = conviction)

- Suppose her signal is *innocent*
- Then *Acquit* is optimal for her if

$$-\Pr(G \mid n-1 \textit{ guilty} \text{ signals and } 1 \textit{ innocent} \text{ signal})(1-z)$$
$$+\Pr(I \mid n-1 \textit{ guilty} \text{ signals and } 1 \textit{ innocent} \text{ signal}) \cdot 0$$

She votes to acquit

Juries

Juror's decision

		other jurors' signals				
		all *innocent*	$n-2$ *innocent*	...	1 *innocent*	all *guilty*
juror i	Acquit	A	A	...	A	A
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Outcome (A = acquittal, C = conviction)

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$$\geq \Pr(G \mid n-1 \textit{ guilty} \text{ signals and 1 } \textit{innocent} \text{ signal}) \cdot 0$$
$$-\Pr(I \mid n-1 \textit{ guilty} \text{ signals and 1 } \textit{innocent} \text{ signal})z$$

She votes to convict

Juries

Juror's decision

		other jurors' signals				
		all *innocent*	$n-2$ *innocent*	...	1 *innocent*	all *guilty*
juror i	Acquit	A	A	...	A	A
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Outcome (A = acquittal, C = conviction)

- or

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Juries

Juror's decision

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Outcome (A = acquittal, C = conviction)

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$$-\Pr(G \mid n-1 \textit{ guilty} \text{ signals and } 1 \textit{ innocent} \text{ signal})(1-z) \geq -\Pr(I \mid n-1 \textit{ guilty} \text{ signals and } 1 \textit{ innocent} \text{ signal})z$$

$$\Leftrightarrow$$

$$\Pr(G \mid n-1 \textit{ guilty} \text{ signals and } 1 \textit{ innocent} \text{ signal})(1-z) \leq (1-\Pr(G \mid n-1 \textit{ guilty} \text{ signals and } 1 \textit{ innocent} \text{ signal}))z$$

Juries

Juror's decision

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Juries

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$$\Pr(G \mid n-1 \textit{ guilty} \text{ signals and } 1 \textit{ innocent} \text{ signal}) \leq z$$

$$\Leftrightarrow \quad \frac{(1-p)p^{n-1}\pi}{(1-p)p^{n-1}\pi + q(1-q)^{n-1}(1-\pi)} \leq z$$

Juries

Juror's decision

other jurors' signals

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Acquit	A	A	...	A	A
Convict	A	A	...	A	C

Outcome (A = acquittal, C = conviction)

- or

Pr($n-1$ *guilty* signals and 1 *innocent* signal | G) $\leq z$

$$\Leftrightarrow \quad \frac{(1-p)p^{n-1}\pi}{(1-p)p^{n-1}\pi + q(1-q)^{n-1}(1-\pi)} \leq z$$

Juries

Juror's decision

		other jurors' signals				
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$$\Leftrightarrow \quad \frac{(1-p)p^{n-1}\pi}{(1-p)p^{n-1}\pi + q(1-q)^{n-1}(1-\pi)} \leq z$$

$$\Leftrightarrow \quad \frac{1}{1 + \frac{q}{1-p}\left(\frac{1-q}{p}\right)^{n-1}\frac{1-\pi}{\pi}} \leq z$$

Juries

Juror's decision

		other jurors' signals				
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Outcome (A = acquittal, C = conviction)

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$$\Pr(G \mid n-1 \textit{ guilty} \text{ signals and 1 } \textit{innocent} \text{ signal}) \leq z$$

$\Leftrightarrow$ $p > \frac{1}{2}$ and $q > \frac{1}{2}$, so $1-q < p$ and hence $((1-q)/p)^{n-1} \to 0$ as $n \to \infty$, so LHS $\to 1$

$$\Leftrightarrow \quad \frac{1}{1+\frac{q}{1-p}\left(\frac{1-q}{p}\right)^{n-1}\frac{1-\pi}{\pi}} \leq z$$

Juries

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- Conclusion: given $z < 1$, for n large enough, juror with *innocent* signal optimally votes *Convict*

Juries

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Juries

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- n may not have to be very large: if $p = q = 0.8$, $\pi = 0.5$, and $n = 12$, LHS of inequality exceeds 0.999999

Juries

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Outcome (A = acquittal, C = conviction)

- Conclusion: given $z < 1$, for n large enough, juror with *innocent* signal optimally votes *Convict*
- Thus for n large enough, every juror's voting according to her signal is *not* a Nash equilibrium
- n may not have to be very large: if $p = q = 0.8$, $\pi = 0.5$, and $n = 12$, LHS of inequality exceeds 0.999999
- If juror with *innocent* signal optimally votes *Convict*, then so does juror with *guilty* signal

Juries

Conclusion

- If all other jurors vote according to their signals, the remaining juror should vote for *conviction* regardless of her signal

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- So there is no equilibrium in which all jurors vote according to their signals

Juries

Conclusion

- If all other jurors vote according to their signals, the remaining juror should vote for *conviction* regardless of her signal
- So there is no equilibrium in which all jurors vote according to their signals
- Note that we have not determined what *is* an equilibrium