Economics 2030

Fall 2018 — Martin J. Osborne

Solutions to Problem Set 2

1. In a second-price auction, the payoff of each player i is $v_i - b_j$ if her bid b_i is equal to the highest bid and b_j is the highest of the other players' bids (possibly equal to b_i) and no player with a lower index submits this bid, and 0 otherwise.

   To show that for any player i the bid $b_i = v_i$ weakly dominates any other bid, let x_i be another bid of player i.

   - If $\max_{j \neq i} b_j \geq v_i$ then if player i bids v_i she guarantees herself a payoff of 0, while by bidding x_i she either does not obtain the object or receives a nonpositive payoff.
   - If $\max_{j \neq i} b_j < v_i$ then if player i bids v_i she obtains the good at the price $\max_{j \neq i} b_j$, while by bidding x_i either she wins and pays the same price or loses.

   Thus player i's payoff from the bid v_i is at least her payoff from another other action. To complete the argument that v_i weakly dominates any other bid, we need to show that for any other bid x_i there are actions of the other players for which the payoff from bidding v_i is higher than the payoff from bidding x_i. If $x_i \neq v_i$, the payoff from bidding v_i is higher than the payoff from bidding x_i when the highest of the other players' bids is between x_i and v_i.

   Thus the bid v_i of player i weakly dominates every other bid.

   An equilibrium in which player j obtains the good is that in which $b_1 < v_j$, $b_j > v_1$, and $b_i = 0$ for all players $i \notin \{1, j\}$.

2. (a) Suppose that all players other than 1 choose the number 1. If player 1 chooses any number from 2 to K, she loses. Thus no action from 2 to K strictly dominates any action.

   I now argue that the action 1 does not strictly dominate any action. Suppose that one of the other players announces 2 and each of the remaining players announces K. I claim that if player 1 announces 1, she loses. To demonstrate this claim, note that for the action profile $(1, 2, K, \ldots, K)$, two-thirds of the average is $\frac{2}{3}K - (\frac{2}{3}K - 1)(2/n)$, which is increasing in n (the number of players). Thus two-thirds of the average is smallest when $n = 3$, in which case it is $\frac{2}{9}K + \frac{2}{3}$. This number is smallest when $K = 4$, in which case it is $\frac{14}{9}$, which is closer to 2 than it is to 1. Thus for any $n \geq 3$ and $K \geq 4$, announcing 1 loses when one of the other players announces 2 and each of the remaining players announces K.

Hence the action of announcing 1 does not strictly dominate any other action.

(b) I claim that the action K is strictly dominated by the action $K - 1$. First I argue that two-thirds of the average of K and $n - 1$ numbers from 1 to K is less than $K - \frac{1}{2}$. The highest value possible for this average is $\frac{2}{3} \cdot K$, which is less than $K - \frac{1}{2}$ if $K > \frac{3}{2}$. Given this fact, if a player announcing K deviates to $K - 1$ then regardless of the other players' announcements, she prefers the resulting action profile: given that two-thirds of the average of the announcements is less than $K - \frac{1}{2}$, $K - 1$ is closer to two-thirds of the average of $K - 1$ and the other players' announcements than K is to two-thirds of the average of of K and the other players' announcements.

Given that $K - 1$ strictly dominates K, K can be eliminated. In the reduced game, $K - 2$ strictly dominates $K - 1$, so that $K - 1$ can be eliminated. Continuing this process, only 1 remains. Thus the only possible Nash equilibrium of the game is the action profile in which every player announces 1.

3. The set of actions $[0, \alpha]$ of each player is nonempty, compact, and convex.

The payoff function of firm i is

$$u_i(q) = q_i \max\left\{\alpha - q_i - \sum_{j \neq i} q_j, 0\right\} - cq_i,$$

which is continuous. (Note that we can write the inverse demand function as $\max\{\alpha - Q, 0\}$.) Thus the preference relation that this payoff function represents is continuous (see, for example, Exercise 3.C.2 in Mas-Colell, Whinston, and Green).

To show that u_i (and hence the preference relation that it represents) is quasiconcave on A_i $(= [0, \alpha])$, we need to show that for any profile

$\overline{q}$ of actions, the set

$$\{q_i \in [0, \alpha] : u_i(\overline{q}_{-i}, q_i) \geq u_i(\overline{q})\}$$

is convex. This set is the set of all numbers q_i such that

$$q_i \max\left\{\alpha - q_i - \sum_{j \neq i} \overline{q}_j, 0\right\} - cq_i \geq u_i(\overline{q}).$$

The function on the left-hand side of this inequality is concave for $0 \leq q_i \leq \alpha - \sum_{j \neq i} \overline{q}_j$ (it is a quadratic in q_i with negative second derivative on this interval), is continuous, and is decreasing for $q_i > \alpha - \sum_{j \neq i} \overline{q}_j$ (it is equal to $-cq_i$ on this interval). Thus for any value of $\overline{q}$, the set of numbers q_i that satisfy the inequality is a (possibly empty) interval, and in particular is convex. Thus u_i is quasiconcave on A_i.

4. The top game is not equivalent, by the following argument. Using either player's payoffs, for equivalence we need α and $\beta > 0$ such that

$$0 = \alpha + \beta \cdot 0, 2 = \alpha + \beta \cdot 1, 3 = \alpha + \beta \cdot 3, \text{and } 4 = \alpha + \beta \cdot 4.$$

From the first equation we have $\alpha = 0$ and hence from the second we have $\beta = 2$. But these values do not satisfy the last two equations. (Alternatively, note that in the *Prisoner's Dilemma* in Figure 17.1, player 1 is indifferent between (*Confess*, *Confess*) and the lottery in which (*Don't confess*, *Confess*) occurs with probability $\frac{2}{3}$ and (*Don't confess*, *Don't confess*) occurs with probability $\frac{1}{3}$, while in the left-hand game she is not.)

The bottom game is equivalent, by the following argument. For the equivalence of player 1's payoffs, we need α and $\beta > 0$ such that

$$0 = \alpha + \beta \cdot 0, 3 = \alpha + \beta \cdot 1, 9 = \alpha + \beta \cdot 3, \text{and } 12 = \alpha + \beta \cdot 4.$$

The first two equations yield $\alpha = 0$ and $\beta = 3$; these values satisfy the second two equations. A similar argument for player 2's payoffs yields $\alpha = -4$ and $\beta = 2$.

5. The best response functions for the left game are shown in the left panel of Figure 1. We see that the game has a unique mixed strategy Nash equilibrium $((\frac{1}{4}, \frac{3}{4}), (\frac{2}{3}, \frac{1}{3}))$.

The best response functions for the right game are shown in the right panel of Figure 1. We see that the mixed strategy Nash equilibria are $((0,1),(1,0))$ and any $(\{(p, 1-p)\}, (0,1))$ with $\frac{1}{2} \leq p \leq 1$.

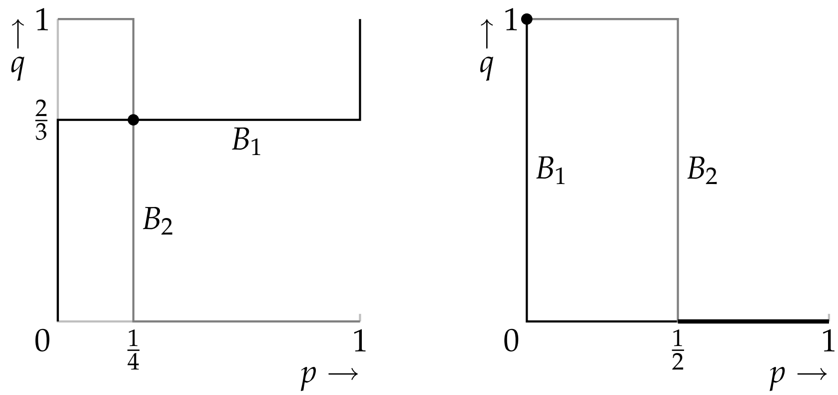

Figure 1. The players' best response functions in the left game (left panel) and right game (right panel) in Exercise 6. The probability that player 1 assigns to T is p and the probability that player 2 assigns to L is q. The disks and the heavy line indicate Nash equilibria.