Economics 2030

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Solutions to Problem Set 1

1. (a) Firm 1's payoff function is

$$\begin{cases} q_1(\alpha - c - q_1 - q_2 - \cdots - q_n) & \text{if } q_1 + q_2 + \cdots + q_n \le \alpha \\ -cq_1 & \text{if } q_1 + q_2 + \cdots + q_n > \alpha. \end{cases}$$

This function is a quadratic in q_1 where it is positive, and is zero when $q_1 = 0$ and when $q_1 = \alpha - c - q_2 - \cdots - q_n$. Thus firm 1's best response function is

$$b_1(q_{-1}) = \begin{cases} (\alpha - c - q_2 - \cdots - q_n)/2 & \text{if } q_2 + \cdots + q_n \le \alpha - c \\ 0 & \text{if } q_2 + \cdots + q_n > \alpha - c \end{cases}$$

(where q_{-1} stands for the list of the outputs of all the firms except firm 1).

The best response functions of every other firm is the same.

(b) The conditions for $(q_1^*, \ldots, q_n^*)$ to be a Nash equilibrium are

$$\begin{aligned} q_1^* &= b_1(q_{-1}^*) \\ q_2^* &= b_2(q_{-2}^*) \\ &\vdots \\ q_n^* &= b_2(q_{-n}^*) \end{aligned}$$

or, in an equilibrium in which all the firms' outputs are positive,

$$\begin{aligned} q_1^* &= \tfrac{1}{2}(\alpha - c - q_2^* - q_3^* - \cdots - q_n^*) \\ q_2^* &= \tfrac{1}{2}(\alpha - c - q_1^* - q_3^* - \cdots - q_n^*) \\ &\vdots \\ q_n^* &= \tfrac{1}{2}(\alpha - c - q_1^* - q_2^* - \cdots - q_{n-1}^*). \end{aligned}$$

(c) We can write the equations as

$$\begin{aligned}
0 &= \alpha - c - 2q_1^* - q_2^* - \cdots - q_{n-1}^* - q_n^* \\
0 &= \alpha - c - q_1^* - 2q_2^* - \cdots - q_{n-1}^* - q_n^* \\
&\vdots \\
0 &= \alpha - c - q_1^* - q_2^* - \cdots - q_{n-1}^* - 2q_n^*.
\end{aligned}$$

If we subtract the second equation from the first we obtain $0 = -q_1^* + q_2^*$, or $q_1^* = q_2^*$. Similarly subtracting the third equation from the second we conclude that $q_2^* = q_3^*$, and continuing with all pairs of equations we deduce that $q_1^* = q_2^* = \cdots = q_n^*$. Let the common value of the firms' outputs be q^*. Then each equation is $0 = \alpha - c - (n+1)q^*$, so that $q^* = (\alpha - c)/(n+1)$.

In summary, the game has a unique Nash equilibrium, in which the output of every firm i is $(\alpha - c)/(n+1)$.

(d) The price at this equilibrium is $\alpha - n(\alpha - c)/(n+1)$, or $(\alpha + nc)/(n+1)$. As n increases this price decreases, approaching c as n increases without bound: $\alpha/(n+1)$ decreases to 0 and $nc/(n+1)$ decreases to c.

2. The game is defined as follows.

Players $\{1, \ldots, n\}$.

Actions The set of actions of each player i is $[0, \infty)$ (the set of possible bids).

Payoffs The payoff of player i is $v_i - b_i$ if her bid b_i is equal to the highest bid and no player with a lower index submits this bid, and 0 otherwise.

The set of Nash equilibria is the set of profiles b of bids with $b_1 \in [v_2, v_1]$, $b_j \leq b_1$ for all $j \neq 1$, and $b_j = b_1$ for some $j \neq 1$.

That all these profiles are Nash equilibria is easy to verify. To see that there are no other equilibria, first we argue that there is no equilibrium in which player 1 does not obtain the object. Suppose that player $i \neq 1$ submits the highest bid b_i and $b_1 < b_i$. If $b_i > v_2$ then player i's payoff is negative, so that he can increase his payoff by bidding 0. If $b_i \leq v_2$ then player 1 can deviate to the bid b_i and win, increasing his payoff.

Now let the winning bid be b^*. We have $b^* \geq v_2$, otherwise player 2 can change his bid to some value in (v_2, b^*) and increase his payoff.

Also $b^* \leq v_1$, otherwise player 1 can reduce her bid and increase her payoff. Finally, $b_j = b^*$ for some $j \neq 1$ otherwise player 1 can increase her payoff by decreasing her bid.

3. (a) Any action profile $(b_1, \ldots, b_n)$ with the following properties is such an equilibrium:

   - the winning bid is b_1
   - the second-highest bid is at least v_2 and is not submitted by player 2
   - the third-highest bid is less than v_2 and at least v_j, where j is the player who submits the second-highest bid.

   In such an action profile, player 1 wins and pays less than v_2. Denote by p^* the price player 1 pays. If the player who submits the second-highest bid changes her bid then either the outcome does not change or, if her bid exceeds b_1, she wins and pays the price p^*, which is at least her valuation (by the third condition). If any other player deviates either the outcome does not change or, if the deviant's bid exceeds b_1, the deviant wins and pays a price equal to the original second-highest bid, which is at least v_2 and hence at least equal to the deviant's valuation.

   (Note that you are asked only to find *one* equilibrium. An example of an action profile that satisfies the conditions is $(b_1, \ldots, b_n) = (v_1, v_n, v_3, v_2, v_n, \ldots, v_n)$.)

   (b) Consider an action profile in which the winner is player n. Player n's bid b_n must be the highest, and the third-highest bid must be at most v_n, otherwise player n's payoff is negative so that she can do better by bidding 0. But now consider a deviation by the player submitting the second-highest bid. If she bids more than b_n then she wins and the price she pays is at most v_n, so her payoff increases. Hence no such action profile is a Nash equilibrium.

4. The best response of player i to c_j is the value of c_i that maximizes $v_i\sqrt{c_1 + c_2} - c_i$. This function is strictly concave, so that if its maximizer is positive, this maximizer is the solution of the first-order condition

$$\tfrac{1}{2}v_i(c_1 + c_2)^{-1/2} - 1 = 0.$$

The solution is $c_i = \frac{1}{4}(v_i)^2 - c_j$, where $j = 2$ if $i = 1$, and $j = 1$ if $i = 2$. This solution is positive if $c_j < \frac{1}{4}(v_i)^2$. If $c_j \geq \frac{1}{4}(v_i)^2$ then i's payoff is decreasing in c_i, so that i's best response is 0.

In summary, player i's best response to c_j is

$$b_i(c_j) = \begin{cases} 0 & \text{if } c_j \geq \frac{1}{4}(v_i)^2 \\ \frac{1}{4}(v_i)^2 - c_j & \text{if } c_j < \frac{1}{4}(v_i)^2 \end{cases}$$

We deduce (draw a diagram of the best response functions) that for any $v_1 \neq v_2$ the game has a unique Nash equilibrium:

$$\begin{cases} (\frac{1}{4}(v_1)^2, 0) & \text{if } v_1 > v_2 \\ (0, \frac{1}{4}(v_2)^2) & \text{if } v_1 < v_2. \end{cases}$$

5. The game may be specified as follows.

   Players $N = \{1, \ldots, n\}$.

   Actions $A_i = [0, 1] \cup \{Out\}$ for all $i \in N$.

   Preferences $a \succ_i a'$ if i obtains more votes than any other player in a and ties with one or more players for the largest number of votes in a', or if she ties with one or more players for the largest number of votes in a and $a'_i = Out$, or if $a_i = Out$ and she loses in a'.

   Let F be the distribution function of the citizens' favorite positions and let $m = F^{-1}(\frac{1}{2})$ be its median (which is unique, since the density f is everywhere positive).

   It is easy to check that for $n = 2$ the game has a unique Nash equilibrium, in which both players choose m.

   The argument that for $n = 3$ the game has no Nash equilibrium is as follows.

   - There is no equilibrium in which some player becomes a candidate and loses, since that player could instead stay out of the competition. Thus in any equilibrium all candidates must tie for first place.
   - There is no equilibrium in which a single player becomes a candidate, since by choosing the same position any of the remaining players ties for first place.
   - There is no equilibrium in which two players become candidates, since by the argument for $n = 2$ in any such equilibrium they must both choose the median position m, in which case the third player can enter close to that position and win outright.

- There is no equilibrium in which all three players become candidates:
  - if all three choose the same position then any one of them can choose a position slightly different and win outright rather than tying for first place;
  - if two choose the same position while the other chooses a different position then the lone candidate can move closer to the other two and win outright.
  - if all three choose different positions then (given that they tie for first place) either of the extreme candidates can move closer to his neighbor and win outright.