ECO2030: Microeconomic Theory II, module 1

Lecture 4

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Bayesian games

Motivational example

Definition

Nash equilibrium
- Example: Cournot’s model
- Example: public good provision with uncertain costs
- Example: exchange game
- Example: information about knowledge

Purification of mixed strategy equilibria

Bayesian games

- Strategic game models situation in which each player knows preferences of other players
- In some situations, players are not certain of other players' preferences
- Model of Bayesian Game allows players to face uncertainty about other players' preferences

Bayesian games: motivational example

Variant of *BoS* with imperfect information

- Player 1 doesn't know whether
  - player 2 prefers to go out with her—player 2 is type m
  - or prefers to avoid her—player 2 is type v
- She thinks probabilities of states are $\frac{1}{2}$–$\frac{1}{2}$
- Player 2 knows player 1's preferences
- Probabilities are involved, so need players' preferences over lotteries, even if interested only in pure strategy equilibria $\Rightarrow$ Bernoulli payoffs

1

2: m

	B	S
B	$2, 1$	$0, 0$
S	$0, 0$	$1, 2$

meet ($\frac{1}{2}$)

2: v

	B	S
B	$2, 0$	$0, 2$
S	$0, 1$	$1, 0$

avoid ($\frac{1}{2}$)

Bayesian games: motivational example

Variant of *BoS* with imperfect information

1

2: m

	B	S
B	$2, 1$	$0, 0$
S	$0, 0$	$1, 2$

meet $(\frac{1}{2})$

2: v

	B	S
B	$2, 0$	$0, 2$
S	$0, 1$	$1, 0$

avoid $(\frac{1}{2})$

An equilibrium

- Player 1 chooses B
- Type m of player 2 chooses B and type v chooses S
- Argument:
  - P1 chooses $B \Rightarrow$ payoff $\frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 0 = 1$; deviates to $S \Rightarrow$ payoff $\frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 1 = \frac{1}{2}$
  - Type m of player 2: deviate to $S \Rightarrow$ payoff 0
  - Type v of player 2: deviate to $B \Rightarrow$ payoff 0

Bayesian games: motivational example

Another variant of *BoS* with imperfect information

Neither player knows whether other wants to go out with her

1: m_1

	B	S
B	2, 1	0, 0
S	0, 0	1, 2

State mm ($\frac{1}{3}$)

	B	S
B	2, 0	0, 2
S	0, 1	1, 0

State mv ($\frac{1}{3}$)

2: m_2

2: v_2

1: v_1

	B	S
B	0, 1	2, 0
S	1, 0	0, 2

State vm ($\frac{1}{6}$)

	B	S
B	0, 0	2, 2
S	1, 1	0, 0

State vv ($\frac{1}{6}$)

Bayesian games: motivational example

Another variant of *BoS* with imperfect information

- Each player receives signal about state before choosing action
- Player i who receives signal t_i is type t_i of player i
- Given prior belief and signal, each type of each player calculates posterior belief

Definition

Elements new relative to strategic game are indicated in red

A Bayesian game consists of

- a finite set N (*players*)
- a set Ω (*states*)
- for each player $i \in N$
  - a set A_i (*actions*)
  - a set T_i (of *signals* that i may receive) and a function $\tau_i : \Omega \to T_i$ that associates a signal with each state (i's signal function)
  - a probability measure p_i on Ω (i's *prior belief*) with $p_i(\tau_i^{-1}(t_i)) > 0$ for all $t_i \in T_i$
  - a *preference relation* over probability distributions over $A \times \Omega$ (represented by the expected value of a Bernoulli payoff function).

Notes

- i has no information: $\tau_i(\omega) = \tau_i(\omega')$ for all ω, ω'
- i has perfect information: $\tau_i(\omega) \neq \tau_i(\omega')$ if $\omega \neq \omega'$

First example

Players $N = \{1, 2\}$ (the pair of people)

States $\Omega = \{meet, avoid\}$

Actions $A_1 = A_2 = \{B, S\}$

Signals $T_1 = \{z\}$ and $\tau_1(meet) = \tau_1(avoid) = z$
$T_2 = \{m, v\}$ and $\tau_2(meet) = m$ and $\tau_2(avoid) = v$

Beliefs $p_1(meet) = p_2(meet) = \frac{1}{2}$,
$p_1(avoid) = p_2(avoid) = \frac{1}{2}$

Payoffs The payoffs $u_i(a, meet)$ of each player i for all possible action pairs are given in the left panel of the figure on the earlier slide and the payoffs $u_i(a, avoid)$ are given in the right panel

Second example

Players $N = \{1, 2\}$ (the pair of people)

States $\Omega = \{mm, mv, vm, vv\}$

Actions $A_1 = A_2 = \{B, S\}$

Signals $T_1 = \{m_1, v_1\}$, $\tau_1(mm) = \tau_1(mv) = m_1$, and $\tau_1(vm) = \tau_1(vv) = v_1$
$T_2 = \{m_2, v_2\}$, $\tau_2(mm) = \tau_2(vm) = m_2$, and $\tau_2(mv) = \tau_2(vv) = v_2$

Beliefs $p_i(mm) = p_i(mv) = \frac{1}{3}$ and $p_i(vm) = p_i(vv) = \frac{1}{6}$ for $i = 1, 2$

Payoffs The payoffs $u_i(a, \omega)$ of each player i for all possible action pairs and states are given on the earlier slide

Second example: Nash equilibria

1: m_1

	B	*S*
B	2, 1	0, 0
S	0, 0	1, 2

State mm ($\frac{1}{3}$)

Posterior: $\frac{1}{2}$

	B	*S*
B	2, 0	0, 2
S	0, 1	1, 0

State mv ($\frac{1}{3}$)

Posterior: $\frac{1}{2}$

Payoffs

$\frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 0 = 1$

$\frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 1 = \frac{1}{2}$

2: m_2

2: v_2

1: v_1

	B	*S*
B	0, 1	2, 0
S	1, 0	0, 2

State vm ($\frac{1}{6}$)

Posterior: $\frac{1}{2}$

	B	*S*
B	0, 0	2, 2
S	1, 1	0, 0

State vv ($\frac{1}{6}$)

Posterior: $\frac{1}{2}$

$\frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 2 = 1$

$\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 0 = \frac{1}{2}$

Payoffs: 1 0 0 2

Nash equilibrium: $((B, B), (B, S))$

Second example: Nash equilibria

1: m_1

	B	S
B	$2, 1$	$0, 0$
S	$0, 0$	$1, 2$

State mm ($\frac{1}{3}$)

Posterior: $\frac{2}{3}$

	B	S
B	$2, 0$	$0, 2$
S	$0, 1$	$1, 0$

State mv ($\frac{1}{3}$)

Posterior: $\frac{2}{3}$

2: m_2

2: v_2

1: v_1

	B	S
B	$0, 1$	$2, 0$
S	$1, 0$	$0, 2$

State vm ($\frac{1}{6}$)

Posterior: $\frac{1}{3}$

	B	S
B	$0, 0$	$2, 2$
S	$1, 1$	$0, 0$

State vv ($\frac{1}{6}$)

Posterior: $\frac{1}{3}$

Another Nash equilibrium: $((S, B), (S, S))$

Nash equilibrium

i's prior & signal

↓

posterior belief about state

action of each type of every other player

↓

actions chosen by each player in every state

↘ ↙

expected payoff of i to each of her actions

Definition

A Nash equilibrium of a Bayesian game is a collection of actions $a(i, t_i)$, one for each type t_i of each player i, such that, for each type t_i of each player i,

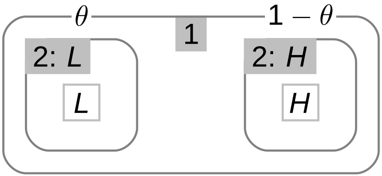
$$a(i, t_i) \text{ maximizes } (i, t_i)\text{'s expected payoff}$$

given the actions $a(j, t_j)$ of every type t_j of every other player j and (i, t_i)'s posterior belief over the set of states.

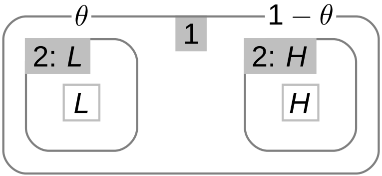
Cournot's duopoly game with imperfect information

Variant of Cournot's duoploy game in which firm 1 does not know firm 2's unit cost

- Both firms produce the good at constant unit cost
- Both firms know that firm 1's unit cost is c
- Firm 2 knows its own unit cost
- Firm 1 believes that firm 2's unit cost is c_L with probability θ and c_H with probability $1 - \theta$, where $0 < \theta < 1$ and $c_L < c_H$



Cournot's duopoly game with imperfect information



Players $N = \{1, 2\}$ (the firms)

States $\Omega = \{L, H\}$

Actions $A_1 = A_2 = \mathbb{R}_+$

Signals $T_1 = \{z\}$, $\tau_1(L) = \tau_1(H) = z$
$T_2 = \{\ell, h\}$, $\tau_2(L) = \ell$, and $\tau_2(H) = h$

Beliefs $p_i(L) = \theta$, $p_i(H) = 1 - \theta$, $i = 1, 2$

Payoffs For $\omega \in \Omega$ we have

$$u_1((q_1, q_2), \omega) = q_1 P(q_1 + q_2) - q_1 c$$
$$u_2((q_1, q_2), \omega) = q_2 P(q_1 + q_2) - q_2 c_\omega$$

Cournot's duopoly game with imperfect information

Nash equilibrium: $(q_1^*, (q_L^*, q_H^*))$ such that

$$q_1^* \text{ maximizes } \theta q_1 P(q_1 + q_L^*) + (1-\theta) q_1 P(q_1 + q_H^*) - c q_1$$

and

$$q_L^* \text{ maximizes } q_L P(q_1^* + q_L) - q_L c_L$$

and

$$q_H^* \text{ maximizes } q_H P(q_1^* + q_H) - q_H c_H$$

Compute best response functions and solve

$$\begin{aligned} q_1^* &= b_1(q_L^*, q_H^*) \\ q_L^* &= b_L(q_1^*) \\ q_H^* &= b_H(q_1^*) \end{aligned}$$

for Nash equilibrium $(q_1^*, (q_L^*, q_H^*))$

Public good provision with uncertain costs

- n people simultaneously decide whether to contribute to the provision of a public good
- The good is provided if and only if at least one person contributes
- Person i's payoff:

$$\begin{cases} 1 - c_i & \text{if } i \text{ contributes} \\ 1 & \text{if } i \text{ does not contribute but good is provided} \\ 0 & \text{if } i \text{ does not contribute and good is not provided} \end{cases}$$

- Each person i knows the value of c_i but not the values of c_j for $j \neq i$
- For each $j \neq i$, person i believes that c_j is distributed independently of c_k for $k \neq j$, according to the continuous cumulative distribution function G on $\mathbb{R}_+$ with $G(0) = 0$

Public good provision with uncertain costs

Bayesian game

Players $\{1, \ldots, n\}$

States $\mathbb{R}^n_+$ (the set of profiles $(c_1, \ldots, c_n)$ of nonnegative numbers)

Actions $\{0, 1\}$ for each player

Signals Set of signals for each player i is $\mathbb{R}_+$ (the set of possible values of c_i); player i's signal function is given by $\tau_i(c) = c_i$ for each $c \in \mathbb{R}^n_+$

Beliefs Each player believes that the probability that $c_i \leq \overline{c}_i$ for each i is $\Pi_{i=1}^n G(c_i)$

Payoffs Payoff of player i for the action profile s in state c is

$$\begin{cases} 1 - c_i & \text{if } s_i = 1 \\ 1 & \text{if } s_i = 0 \text{ and } s_j = 1 \text{ for some } j \neq i \\ 0 & \text{if } s_j = 0 \text{ for all } j \end{cases}$$

Public good provision with uncertain costs

Nash equilibrium

- Seems reasonable that game has equilibrium in which each player contributes if and only if her cost is low
- Check for (symmetric pure) equilibrium in which each player j contributes if and only if $c_j \leq \overline{c}$, for some number $\overline{c}$
- Suppose every player $j \neq i$ uses this strategy
- Then probability that at least one of these players contributes is $1 - (1 - G(\overline{c}))^{n-1}$
- So player i's payoff

$$\begin{cases} 1 - c_i & \text{if she contributes} \\ 1 - (1 - G(\overline{c}))^{n-1} & \text{is she does not contribute} \end{cases}$$

Public good provision with uncertain costs

Nash equilibrium

- Player i's payoff:

$$\begin{cases} 1 - c_i & \text{if she contributes} \\ 1 - (1 - G(\overline{c}))^{n-1} & \text{if she does not contribute} \end{cases}$$

- For strategy profile to be equilibrium, we want contribution by i to be optimal if $c_i \leq \overline{c}$ and non-contribution to be optimal if $c_i \geq \overline{c}$
- That is, want

$$\begin{cases} 1 - c_i \geq 1 - (1 - G(\overline{c}))^{n-1} & \text{if } c_i \leq \overline{c} \\ 1 - c_i \leq 1 - (1 - G(\overline{c}))^{n-1} & \text{if } c_i \geq \overline{c} \end{cases}$$

- Conditions are satisfied if and only if

$$1 - \overline{c} = 1 - (1 - G(\overline{c}))^{n-1}$$

or

$$\overline{c} = (1 - G(\overline{c}))^{n-1}$$

Public good provision with uncertain costs

Nash equilibrium

- That is, if strategy of every player $j \neq i$ satisfies

$$\begin{cases} \text{contribute} & \text{if } c_j < \overline{c} \\ \text{don't contribute} & \text{if } c_j > \overline{c} \end{cases}$$

where $\overline{c} = (1 - G(\overline{c}))^{n-1}$, then it is optimal for player i to use strategy satisfying these conditions

- Hence strategy profile in which each player uses such a strategy is Nash equilibrium
- Does such a value of $\overline{c}$ exist?
- Function $\overline{c} - (1 - G(\overline{c}))^{n-1}$ is continuous and has values

$$\begin{cases} -1 & \text{for } \overline{c} = 0 \\ > 0 & \text{for } \overline{c} \text{ large enough} \end{cases}$$

- Thus value of $\overline{c}$ exists for which $\overline{c} = (1 - G(\overline{c}))^{n-1}$

Public good provision with uncertain costs

Summary

Bayesian game has (pure strategy) Nash equilibrium in which the strategy of every player i satisfies

$$\begin{cases} \text{contribute} & \text{if } c_i < \overline{c} \\ \text{don't contribute} & \text{if } c_i > \overline{c} \end{cases}$$

where

$$\overline{c} = (1 - G(\overline{c}))^{n-1}$$

Exchange game

- Each of two players receives a ticket on which there is a number in some finite subset V of the interval $[0, 1]$
- The number on a player's ticket is the size of a prize that she may receive
- Each player knows sees her ticket, but not any other player's ticket
- Each player believes that the prizes are drawn independently from the same distribution F (which assigns positive probability to each possible prize)
- Each player is asked independently and simultaneously whether she wants to exchange her prize for the other player's prize
- If both players agree then the prizes are exchanged; otherwise each player receives her own prize
- Each player's objective is to maximize her expected payoff

Exchange game

Bayesian game

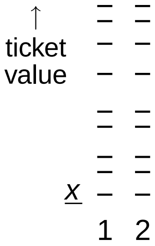
Players	$\{1, 2\}$
States	$V \times V$ (set of pairs of ticket values)
Actions	$\{\text{exchange}, \text{don't exchange}\}$ for each player
Signals	Set of signals for each player i is V; player i's signal function is $\tau_i(s_1, s_2) = s_i$
Beliefs	Each player's belief is that s_1 and s_2 are two independent draws from F
Payoffs	Payoff of player i for the action profile s in state c is

$$u_i((a_1, a_2), \omega) = \begin{cases} \omega_j & \text{if } a_1 = a_2 = \textit{Exchange} \\ \omega_i & \text{otherwise} \end{cases}$$

Exchange game

Nash equilibrium

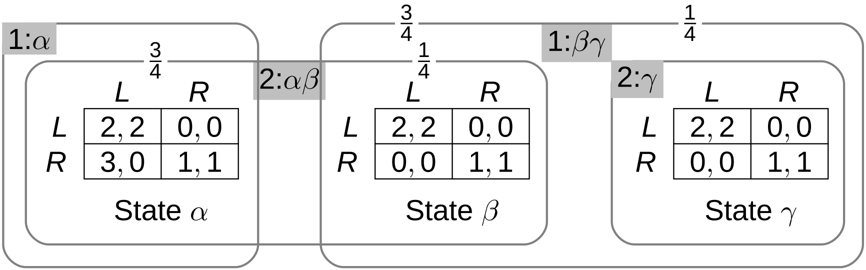
In a Nash equilibrium, which tickets are exchanged?



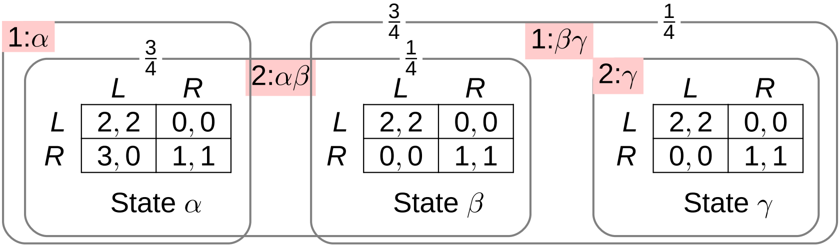
- Let $\underline{x}$ be smallest possible prize and let M_i be highest type of player i that chooses *Exchange*
- If $M_i \geq M_j$ and $M_i > \underline{x}$ then type M_i of player i does not optimally choose *Exchange*, since expected value of prizes of types of player j that choose *Exchange* is less than M_i
- Thus in any Nash equilibrium $M_i = M_j = \underline{x}$: the only prizes that may be exchanged are the smallest

Example: Imperfect information about knowledge

Bayesian game may be used to model not only situations in which players are uncertain about each others' preferences, but also situations in which they are uncertain about each others' *knowledge*.



Example: Imperfect information about knowledge



State α

	L	R
L	$2,2$	$0,0$
R	$3,0$	$1,1$

State β

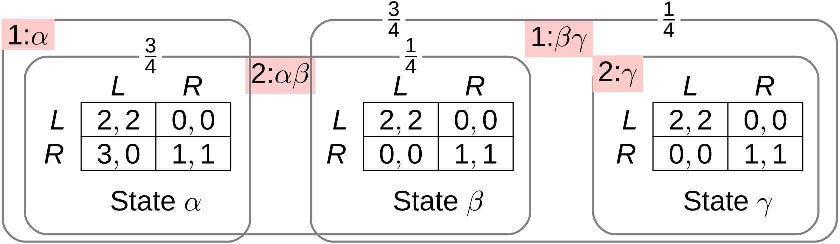
	L	R
L	$2,2$	$0,0$
R	$0,0$	$1,1$

State γ

	L	R
L	$2,2$	$0,0$
R	$0,0$	$1,1$

- Player 2's preferences same in all three states; player 1's preferences same in states β and γ.
- In state γ:
  - 1 knows 2's preferences (which are same in all states)
  - 2 knows 1's preferences
  - 2 knows that 1 knows 2's preferences (2 knows state is γ, and hence knows 1 knows state is either β or γ)
  - 1 does not know that 2 knows 1's preferences: 1 knows only that state is either β or γ, and in state β player 2 does not know whether state is α or β, and hence does not know 1's preferences (because 1's preferences in α and β differ)

Example: Imperfect information about knowledge



	L	R
L	2, 2	0, 0
R	3, 0	1, 1

State α

	L	R
L	2, 2	0, 0
R	0, 0	1, 1

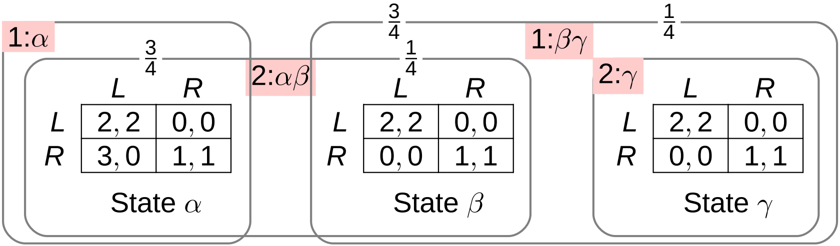
State β

	L	R
L	2, 2	0, 0
R	0, 0	1, 1

State γ

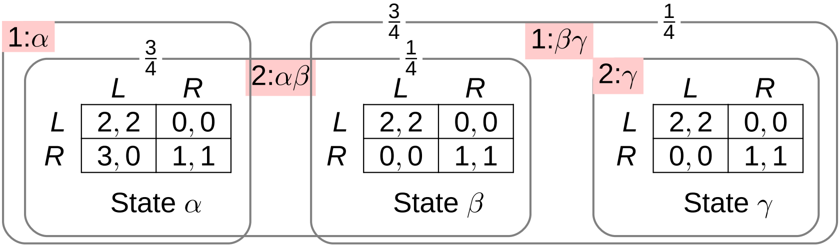
- This imperfection in player 1's knowledge of player 2's information significantly affects the equilibria of the game:
  - If information were perfect in state γ, then both (L, L) and (R, R) would be Nash equilibria.
  - However, whole game has *unique* Nash equilibrium, in which outcome in state γ is (R, R). The incentives faced by player 1 in state α "infect" the remainder of the game.

Example: Imperfect information about knowledge



- In any Nash equilibrium, action of type α of player 1 is *R*, because *R* strictly dominates *L*
- Consider type $\alpha\beta$ of player 2:
  - type $\beta\gamma$ of 1 chooses *L* $\Rightarrow$ expected payoff of type $\alpha\beta$ of player 2 to *L* is $\frac{3}{4} \cdot 0 + \frac{1}{4} \cdot 2 = \frac{1}{2}$ and to *R* is $\frac{3}{4} \cdot 1 + \frac{1}{4} \cdot 0 = \frac{3}{4}$
  - type $\beta\gamma$ of 1 chooses *R* $\Rightarrow$ expected payoff of type $\alpha\beta$ of player 2 to *L* is 0 and to *R* is 1
  - Thus in any Nash equilibrium, action of type $\alpha\beta$ of player 2 is *R*

Example: Imperfect information about knowledge



	L	R
L	$2,2$	$0,0$
R	$3,0$	$1,1$

State α

	L	R
L	$2,2$	$0,0$
R	$0,0$	$1,1$

State β

	L	R
L	$2,2$	$0,0$
R	$0,0$	$1,1$

State γ

- Now consider type $\beta\gamma$ of player 1. By same argument as before, her best action is R, regardless of action of type γ of player 2. Thus in any Nash equilibrium, action of type $\beta\gamma$ of player 1 is R.
- Finally, best action of type γ of player 2 is also R

Hence unique Nash equilibrium: $((R,R),(R,R))$.

Example: Imperfect information about knowledge

- Can add states, leading imperfection in information to be arbitrarily minor.
- Still will be unique Nash equilibrium in which all types of all players choose R.

Abstract of Harsanyi (1973)

"Equilibrium points in mixed strategies seem to be unstable, because any player can deviate without penalty from his equilibrium strategy even if he expects all other players to stick to theirs. This paper proposes a model under which most mixed-strategy equilibrium points have full stability. It is argued that for any game Γ the players' uncertainty about the other players' exact payoffs can be modeled as a disturbed game Γ^*, i.e., as a game with small random fluctuations in the payoffs. Any equilibrium point in Γ, whether it is in pure or in mixed strategies, can "almost always" be obtained as a limit of a pure-strategy equilibrium point in the corresponding disturbed game Γ^* when all disturbances go to zero. Accordingly, mixed-strategy equilibrium points are stable — even though the players may make no deliberate effort to use their pure strategies with the probability weights prescribed by their mixed equilibrium strategies — because the random fluctuations in their payoffs will make them use their pure strategies approximately with the prescribed probabilities."

Purification of mixed strategy equilibria

	Bach	*Stravinsky*
Bach	$2+\sigma\varepsilon$,$1+\sigma\eta$	$\sigma\varepsilon, 0$
Stravinsky	0,$\sigma\eta$	$1, 2$

- Three NEs: (B, B), (S, S), and $((\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3}))$
- In mixed strategy equilibrium, each player is indifferent between all her strategies—she has no positive incentive to choose equilibrium strategy
- Suppose that players have “moods” that affect the intensity of their preferences
- Player 1 has type $\varepsilon \sim U[-1, 1]$, unobservable to player 2
- Parameter $\sigma \in (0, 1)$ captures strength of effect of moods
- Player 2 similarly has type $\eta \sim U[-1, 1]$, independent of ε
- We are interested in the outcome of the game when σ is close to zero

Purification of mixed strategy equilibria

	Bach	*Stravinsky*
Bach	$2 + \sigma\varepsilon$,$1 + \sigma\eta$	$\sigma\varepsilon, 0$
Stravinsky	0,$\sigma\eta$	$1, 2$

Bayesian game for given σ

Players 1 and 2

States Set $[-1, 1] \times [-1, 1]$ of pairs of moods

Actions $\{B, S\}$ for each player

Signals $T_1 = [-1, 1]$, $\tau_1(\varepsilon, \eta) = \varepsilon$; $T_2 = [-1, 1]$, $\tau_2(\varepsilon, \eta) = \eta$;

Beliefs ε and η are $U[-1, 1]$ independently

Payoffs Given in table

Purification of mixed strategy equilibria

	Bach	*Stravinsky*
Bach	$2 + \sigma\varepsilon, 1 + \sigma\eta$	$\sigma\varepsilon, 0$
Stravinsky	$0, \sigma\eta$	$1, 2$

$$\varepsilon \in [-1, 1], \sigma \in (0, 1)$$

Nash equilibria

- If every type of player 2 chooses *B*, optimal action of every type of player 1 is *B* (for any σ) and if every type of player 1 chooses *B*, optimal action of every type of player 2 is *B*
- So NE in which every type of each player chooses *B*
- Also NE in which every type of each player chooses *S*

Purification of mixed strategy equilibria

	Bach	Stravinsky
Bach	$2+\sigma\varepsilon, 1+\sigma\eta$	$\sigma\varepsilon, 0$
Stravinsky	$0, \sigma\eta$	$1, 2$

$$\varepsilon \in [-1, 1], \sigma \in (0, 1)$$

Nash equilibria

- Look for equilibrium in which each player chooses *B* when mood is above some threshold, otherwise *S*
- Suppose player 2 chooses *B* if $\eta > \overline{\eta}$, otherwise *S* $\Rightarrow$ player 2 chooses *B* with probability $\frac{1}{2}(1-\overline{\eta})$
- Then for player 1, *B* is a best response if and only if

$$\tfrac{1}{2}(1-\overline{\eta})(2+\sigma\varepsilon)+\tfrac{1}{2}(1+\overline{\eta})\sigma\varepsilon \geq \tfrac{1}{2}(1-\overline{\eta})\cdot 0+\tfrac{1}{2}(1+\overline{\eta})\cdot 1$$

  or $\varepsilon \geq (3\overline{\eta}-1)/2\sigma$

- Player 1 chooses *B* if $\varepsilon > (3\overline{\eta}-1)/2\sigma$, *S* if $\varepsilon < (3\overline{\eta}-1)/2\sigma$

Purification of mixed strategy equilibria

	Bach	*Stravinsky*
Bach	$2+\sigma\varepsilon, 1+\sigma\eta$	$\sigma\varepsilon, 0$
Stravinsky	$0, \sigma\eta$	$1, 2$

$$\varepsilon \in [-1, 1], \sigma \in (0, 1)$$

Nash equilibria

- Similarly, if player 1 chooses *B* if $\varepsilon > \overline{\varepsilon}$ then *B* is a best response for player 2 if and only if $\eta > (1+3\varepsilon)/2\sigma$
- So equilibrium in which

$$\overline{\eta} = (1+3\overline{\varepsilon})/2\sigma$$
$$\overline{\varepsilon} = (3\overline{\eta} - 1)/2\sigma$$

or

$$\overline{\varepsilon} = -\frac{1}{2\sigma+3} \qquad \text{and} \qquad \overline{\eta} = \frac{1}{2\sigma+3}$$

Purification of mixed strategy equilibria

	Bach	*Stravinsky*
Bach	$2+\sigma\varepsilon, 1+\sigma\eta$	$\sigma\varepsilon, 0$
Stravinsky	$0, \sigma\eta$	$1, 2$

$$\varepsilon \in [-1, 1], \sigma \in (0, 1)$$

Nash equilibria

Thus for given value of σ, Bayesian game has Nash equilibrium

- player 1 chooses *B* if and only if
$$\varepsilon > -1/(2\sigma + 3) \rightarrow -\tfrac{1}{3} \text{ as } \sigma \rightarrow 0$$
- player 2 chooses *B* if and only if
$$\eta > 1/(2\sigma + 3) \rightarrow \tfrac{1}{3} \text{ as } \sigma \rightarrow 0$$
- So limit of these (pure, strict) equilibria as $\sigma \rightarrow 0$ is mixed strategy equilibrium of original game (with $\sigma = 0$)
- For any $\sigma > 0$, each type of player has strict incentive to choose equilibrium action

Purification of mixed strategy equilibria

General result

Let $G = \langle N, (A_i), (u_i) \rangle$ be a finite strategic game

- For each $i \in N$ and $a \in A$ let $\varepsilon_i(a)$ be a random variable with range $[-1, 1]$
- Assume each $\varepsilon_i(a)$ is independent of every other
- Assume each $\varepsilon_i(a)$ has an absolutely continuous distribution function ($\Rightarrow$ has density) and its density is continuously differentiable
- Will consider game in which payoff of player i's payoff to a is $u_i(a) + \varepsilon_i(a)$
- Let $\varepsilon = (\varepsilon_i)_{i \in N}$

	L	R
T	$v_1 + 0.1, v_2 - 0.5$	$w_1 - 0.2, w_2 + 0.3$
B	$x_1 - 0.3, x_2 + 0.1$	$y_1 + 0.8, y_2 - 0.1$

Purification of mixed strategy equilibria

General result

Bayesian game $G(\varepsilon)$

Players N

States $[-1, 1]^{N \times A}$ (set of possible values of $\varepsilon_i(a)$'s)

Actions A_i for each player i

Signals Set of signals for each player i is $[-1, 1]^{A_i}$; player i's signal function is $\tau_i(\varepsilon) = \varepsilon_i$

Beliefs The belief of each player i is that each $\varepsilon_i(a)$ is an independent draw from its distribution

Payoffs Payoff of player i for the action profile a in state ω is $u_i(a) + \omega_i(a)$ (where $\omega_i(a)$ is the realization of $\varepsilon_i(a)$)

Purification of mixed strategy equilibria

General result

Proposition (*Harsanyi 1973*)

For almost any finite strategic game G, almost any mixed strategy equilibrium of G is the mixed strategy profile associated with the limit as $\gamma \to 0$ of a sequence of pure strategy equilibria of $G(\gamma\varepsilon)$.

Note that each pure strategy equilibrium of $G(\gamma\varepsilon)$ is strict.

So we can think of mixed strategy equilibria as approximations of strict pure strategy equilibria when players have a small amount of private information about their payoffs.