ECO2030: Microeconomic Theory II, module 1

Lecture 3

Martin J. Osborne

Department of Economics
University of Toronto

2018.10.30

© 2018 by Martin J. Osborne

Table of contents

Finding mixed strategy equilibria
- Finding all mixed strategy equilibria

Strict Nash equilibrium

Example: reporting a crime

Domination by a mixed strategy

Rationalizability

Bayesian games: introduction

Characterization of mixed strategy Nash equilibrium

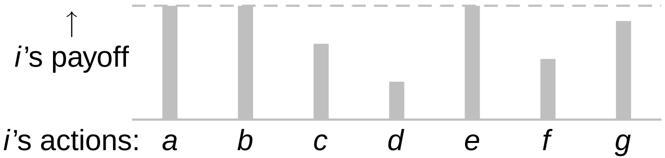
- α^* is a mixed strategy Nash equilibrium $\Leftrightarrow$ α_i^* is a best response to α_{-i}^* for all i

Characterization of mixed strategy Nash equilibrium

- α^* is a mixed strategy Nash equilibrium $\Leftrightarrow$ α_i^* is a best response to α_{-i}^* for all i
- When is a mixed strategy α_i a best response to α_{-i}^*?

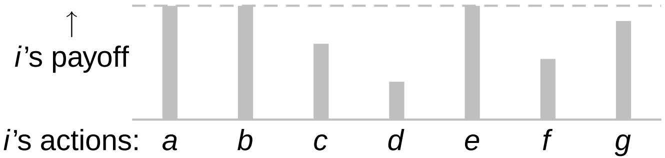
Characterization of mixed strategy Nash equilibrium

- α^* is a mixed strategy Nash equilibrium $\Leftrightarrow$ α_i^* is a best response to α_{-i}^* for all i
- When is a mixed strategy α_i a best response to α_{-i}^*?
- Suppose expected payoffs to player i's actions, given α_{-i}^*, are:



Characterization of mixed strategy Nash equilibrium

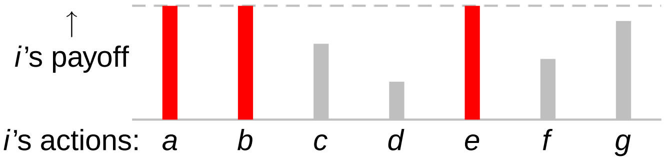
- α^* is a mixed strategy Nash equilibrium $\Leftrightarrow$ α_i^* is a best response to α_{-i}^* for all i
- When is a mixed strategy α_i a best response to α_{-i}^*?
- Suppose expected payoffs to player i's actions, given α_{-i}^*, are:



- What mixed strategies of player i are best responses to α_{-i}^*?

Characterization of mixed strategy Nash equilibrium

- α^* is a mixed strategy Nash equilibrium $\Leftrightarrow$ α_i^* is a best response to α_{-i}^* for all i
- When is a mixed strategy α_i a best response to α_{-i}^*?
- Suppose expected payoffs to player i's actions, given α_{-i}^*, are:



- What mixed strategies of player i are best responses to α_{-i}^*?
- Mixed strategy α_i is a best response to α_{-i}^* if and only if it assigns probability zero to c, d, and f; all probability must be assigned to actions that are best responses to α_{-i}^*

Characterization of mixed strategy Nash equilibrium

Definition

Support of mixed strategy $=$ set of actions to which strategy assigns positive probability

Characterization of mixed strategy Nash equilibrium

Definition

Support of mixed strategy $=$ set of actions to which strategy assigns positive probability

Proposition (*Lemma 33.2*)

α^* is a mixed strategy Nash equilibrium

$$\Leftrightarrow$$

for every player i, α_i^* is a best response to α_{-i}^*

Characterization of mixed strategy Nash equilibrium

Definition

Support of mixed strategy $=$ set of actions to which strategy assigns positive probability

Proposition (*Lemma 33.2*)

α^* is a mixed strategy Nash equilibrium

$\Leftrightarrow$

for every player *i*, α_i^* is a best response to α_{-i}^*

$\Leftrightarrow$

for every player *i*, every action in support of α_i^* is a best response to α_{-i}^*.

Characterization of mixed strategy Nash equilibrium

- Consider two-player game

Characterization of mixed strategy Nash equilibrium

- Consider two-player game
- Actions $1, \ldots, k$ for player 1 and $1, \ldots, m$ for player 2

Characterization of mixed strategy Nash equilibrium

- Consider two-player game
- Actions $1, \ldots, k$ for player 1 and $1, \ldots, m$ for player 2
- Mixed strategy pair $((p_1, \ldots, p_k), (q_1, \ldots, q_m))$ is mixed strategy Nash equilibrium if and only if there exist numbers π_1 and π_2 such that

$$E(u_1(j, q)) \begin{cases} = \pi_1 & \text{for every action } j \text{ with } p_j > 0 \\ \leq \pi_1 & \text{for every action } j \text{ with } p_j = 0 \end{cases}$$

Expected payoff of player 1 when she chooses action j and player 2 chooses *mixed strategy* q

Characterization of mixed strategy Nash equilibrium

- Consider two-player game
- Actions $1, \ldots, k$ for player 1 and $1, \ldots, m$ for player 2
- Mixed strategy pair $((p_1, \ldots, p_k), (q_1, \ldots, q_m))$ is mixed strategy Nash equilibrium if and only if there exist numbers π_1 and π_2 such that

$$E(u_1(j, q)) \begin{cases} = \pi_1 & \text{for every action } j \text{ with } p_j > 0 \\ \leq \pi_1 & \text{for every action } j \text{ with } p_j = 0 \end{cases}$$

and

$$E(u_2(p, j)) \begin{cases} = \pi_2 & \text{for every action } j \text{ with } q_j > 0 \\ \leq \pi_2 & \text{for every action } j \text{ with } q_j = 0 \end{cases}$$

Example

Is strategy pair a mixed strategy Nash equilibrium?

	L (0)	C ($\frac{1}{3}$)	R ($\frac{2}{3}$)
T ($\frac{3}{4}$)	$\cdot, 2$	$3, 3$	$1, 1$
M (0)	$\cdot, \cdot$	$0, \cdot$	$2, \cdot$
B ($\frac{1}{4}$)	$\cdot, 4$	$5, 1$	$0, 7$

(Unspecified payoffs are irrelevant.)

Example

Is strategy pair a mixed strategy Nash equilibrium?

	L (0)	C ($\frac{1}{3}$)	R ($\frac{2}{3}$)	
T ($\frac{3}{4}$)	$\cdot, 2$	$3, 3$	$1, 1$	$\frac{5}{3}$
M (0)	$\cdot, \cdot$	$0, \cdot$	$2, \cdot$	$\frac{4}{3}$
B ($\frac{1}{4}$)	$\cdot, 4$	$5, 1$	$0, 7$	$\frac{5}{3}$
	$\frac{5}{2}$	$\frac{5}{2}$	$\frac{5}{2}$	

(Unspecified payoffs are irrelevant.)

- Compute expected payoff of each action, given other player's actions

Example

Is strategy pair a mixed strategy Nash equilibrium?

	L (0)	C ($\frac{1}{3}$)	R ($\frac{2}{3}$)	
T ($\frac{3}{4}$)	$\cdot, 2$	$3, 3$	$1, 1$	$\frac{5}{3}$
M (0)	$\cdot, \cdot$	$0, \cdot$	$2, \cdot$	$\frac{4}{3}$
B ($\frac{1}{4}$)	$\cdot, 4$	$5, 1$	$0, 7$	$\frac{5}{3}$
	$\frac{5}{2}$	$\frac{5}{2}$	$\frac{5}{2}$	

(Unspecified payoffs are irrelevant.)

- Compute expected payoff of each action, given other player's actions
- If every action in support of each player's mixed strategy yields same payoff *and*

Example

Is strategy pair a mixed strategy Nash equilibrium?

	L (0)	C ($\frac{1}{3}$)	R ($\frac{2}{3}$)	
T ($\frac{3}{4}$)	$\cdot, 2$	$3, 3$	$1, 1$	$\frac{5}{3}$
M (0)	$\cdot, \cdot$	$0, \cdot$	$2, \cdot$	$\frac{4}{3}$
B ($\frac{1}{4}$)	$\cdot, 4$	$5, 1$	$0, 7$	$\frac{5}{3}$
	$\frac{5}{2}$	$\frac{5}{2}$	$\frac{5}{2}$	

(Unspecified payoffs are irrelevant.)

- Compute expected payoff of each action, given other player's actions
- If every action in support of each player's mixed strategy yields same payoff *and* actions outside support yield at most this payoff

Example

Is strategy pair a mixed strategy Nash equilibrium?

	L (0)	C ($\frac{1}{3}$)	R ($\frac{2}{3}$)	
T ($\frac{3}{4}$)	$\cdot, 2$	$3, 3$	$1, 1$	$\frac{5}{3}$
M (0)	$\cdot, \cdot$	$0, \cdot$	$2, \cdot$	$\frac{4}{3}$
B ($\frac{1}{4}$)	$\cdot, 4$	$5, 1$	$0, 7$	$\frac{5}{3}$
	$\frac{5}{2}$	$\frac{5}{2}$	$\frac{5}{2}$	

(Unspecified payoffs are irrelevant.)

- Compute expected payoff of each action, given other player's actions
- If every action in support of each player's mixed strategy yields same payoff *and* actions outside support yield at most this payoff *then* strategy pair is mixed strategy Nash equilibrium

Finding mixed strategy Nash equilibria

Finding mixed strategy equilibrium in which each player's strategy has given support, if one exists:

- For each player $i = 1, \ldots, n$, let $S_i \subseteq A_i$

Finding mixed strategy Nash equilibria

Finding mixed strategy equilibrium in which each player's strategy has given support, if one exists:

- For each player $i = 1, \ldots, n$, let $S_i \subseteq A_i$
- To find mixed strategy equilibrium $(p_1, \ldots, p_n)$ in which support of p_i is S_i for each player i,

Finding mixed strategy Nash equilibria

Finding mixed strategy equilibrium in which each player's strategy has given support, if one exists:

- For each player $i = 1, \ldots, n$, let $S_i \subseteq A_i$
- To find mixed strategy equilibrium $(p_1, \ldots, p_n)$ in which support of p_i is S_i for each player i,
  - find solution of system of equations

$$\begin{aligned} E(u_1(j, p_{-1})) &= \pi_1 \quad \text{for every } j \in S_i \\ &\vdots \\ E(u_n(j, p_{-n})) &= \pi_n \quad \text{for every } j \in S_n \end{aligned}$$

(if one exists)

Finding mixed strategy Nash equilibria

Finding mixed strategy equilibrium in which each player's strategy has given support, if one exists:

- For each player $i = 1, \ldots, n$, let $S_i \subseteq A_i$
- To find mixed strategy equilibrium $(p_1, \ldots, p_n)$ in which support of p_i is S_i for each player i,
  - find solution of system of equations

$$E(u_1(j, p_{-1})) = \pi_1 \quad \text{for every } j \in S_i$$

$$\vdots$$

$$E(u_n(j, p_{-n})) = \pi_n \quad \text{for every } j \in S_n$$

    (if one exists)
  - Check, for each player i, whether $E(u_i(j, p_{-i})) \leq \pi_i$ for every action j of player i

Example

	L	C	R
T	$1, 2$	$3, 3$	$1, 1$
M	$1, 0$	$0, 2$	$2, 1$
B	$3, 4$	$2, 1$	$0, 7$

Example

	L	C	R
T	1, 2	3, 3	1, 1
M	1, 0	0, 2	2, 1
B	3, 4	2, 1	0, 7

Equilibrium in which support of player 1's strategy is $\{T, M\}$ and support of player 2's strategy is $\{L, C\}$?

Example

	L	C	R
T	$1, 2$	$3, 3$	$1, 1$
M	$1, 0$	$0, 2$	$2, 1$
B	$3, 4$	$2, 1$	$0, 7$

Equilibrium in which support of player 1's strategy is $\{T, M\}$ and support of player 2's strategy is $\{L, C\}$?

For player 1 to get same payoff from T and B need

$$\begin{aligned} q_1 + 3q_2 &= \pi_1 \\ q_1 &= \pi_1 \end{aligned}$$

which obviously has no solution with $q_2 > 0$

Example

	L	C	R
T	1, 2	3, 3	1, 1
M	1, 0	0, 2	2, 1
B	3, 4	2, 1	0, 7

Equilibrium in which support of player 1's strategy is $\{T, M\}$ and support of player 2's strategy is $\{L, C\}$?

For player 1 to get same payoff from T and B need

$$\begin{aligned} q_1 + 3q_2 &= \pi_1 \\ q_1 &= \pi_1 \end{aligned}$$

which obviously has no solution with $q_2 > 0$

So no equilibrium with these supports

Example

	L	C	R
T	1, 2	3, 3	1, 1
M	1, 0	0, 2	2, 1
B	3, 4	2, 1	0, 7

Example

	L	C	R
T	$1, 2$	$3, 3$	$1, 1$
M	$1, 0$	$0, 2$	$2, 1$
B	$3, 4$	$2, 1$	$0, 7$

Equilibrium in which support of player 1's strategy is $\{T, B\}$ and support of player 2's strategy is $\{L, C\}$?

Example

	L	C	R
T	$1, 2$	$3, 3$	$1, 1$
M	$1, 0$	$0, 2$	$2, 1$
B	$3, 4$	$2, 1$	$0, 7$

Equilibrium in which support of player 1's strategy is $\{T, B\}$ and support of player 2's strategy is $\{L, C\}$?

For player 1 to get same payoff from T and B need

$$\begin{aligned} q_1 + 3q_2 &= \pi_1 \\ 3q_1 + 2q_2 &= \pi_1 \end{aligned} \quad \Rightarrow q_1 = \tfrac{1}{3}, q_2 = \tfrac{2}{3}, \pi_1 = \tfrac{7}{3} \quad (q_1 + q_2 = 1)$$

Example

	L	C	R
T	$1, 2$	$3, 3$	$1, 1$
M	$1, 0$	$0, 2$	$2, 1$
B	$3, 4$	$2, 1$	$0, 7$

Equilibrium in which support of player 1's strategy is $\{T, B\}$ and support of player 2's strategy is $\{L, C\}$?

For player 1 to get same payoff from T and B need

$$\begin{aligned} q_1 + 3q_2 &= \pi_1 \\ 3q_1 + 2q_2 &= \pi_1 \end{aligned} \qquad \Rightarrow q_1 = \tfrac{1}{3}, q_2 = \tfrac{2}{3}, \pi_1 = \tfrac{7}{3} \quad (q_1 + q_2 = 1)$$

For player 2 to get same payoff from L and C need

$$\begin{aligned} 2p_1 + 4p_3 &= \pi_2 \\ 3p_1 + p_3 &= \pi_2 \end{aligned} \qquad \Rightarrow p_1 = \tfrac{3}{4}, p_3 = \tfrac{1}{4}, \pi_2 = \tfrac{10}{4} \quad (p_1 + p_3 = 1)$$

Example

	L	C	R
T	$1, 2$	$3, 3$	$1, 1$
M	$1, 0$	$0, 2$	$2, 1$
B	$3, 4$	$2, 1$	$0, 7$

Equilibrium in which support of player 1's strategy is $\{T, B\}$ and support of player 2's strategy is $\{L, C\}$?

$$q_1 = \frac{1}{3}, q_2 = \frac{2}{3}, \pi_1 = \frac{7}{3}; p_1 = \frac{3}{4}, p_3 = \frac{1}{4}, \pi_2 = \frac{10}{4}$$

For an equilibrium, need also

$$\text{1's payoff to } M \leq \pi_1 \quad \Rightarrow q_1 \leq \frac{7}{3}, \text{which is true}$$

and

$$\text{2's payoff to } R \leq \pi_2 \quad \Rightarrow p_1 + 7p_3 \leq \frac{10}{4}, \text{which is true}$$

Example

	L	C	R
T	1, 2	3, 3	1, 1
M	1, 0	0, 2	2, 1
B	3, 4	2, 1	0, 7

Equilibrium in which support of player 1's strategy is $\{T, B\}$ and support of player 2's strategy is $\{L, C\}$?

$$q_1 = \frac{1}{3}, q_2 = \frac{2}{3}, \pi_1 = \frac{7}{3}; p_1 = \frac{3}{4}, p_3 = \frac{1}{4}, \pi_2 = \frac{10}{4}$$

For an equilibrium, need also

$$\text{1's payoff to } M \leq \pi_1 \quad \Rightarrow q_1 \leq \frac{7}{3}, \text{which is true}$$

and

$$\text{2's payoff to } R \leq \pi_2 \quad \Rightarrow p_1 + 7p_3 \leq \frac{10}{4}, \text{which is true}$$

So equilibrium exists with these supports

Finding all mixed strategy equilibria

Procedure

- For each player i, choose a set $S_i \subseteq A_i$

Finding all mixed strategy equilibria

Procedure

- For each player i, choose a set $S_i \subseteq A_i$
- Find all the mixed strategy equilibria of the game in which the support of the strategy of each player i is S_i

Finding all mixed strategy equilibria

Procedure

- For each player i, choose a set $S_i \subseteq A_i$
- Find all the mixed strategy equilibria of the game in which the support of the strategy of each player i is S_i
- Repeat for all possible profiles $(S_i)_{i \in N}$ of such subsets

Example

	L	R
T	$1, 1$	$2, 0$
B	$1, 0$	$0, 2$

Example

	L	R
T	$1, 1$	$2, 0$
B	$1, 0$	$0, 2$

- Subsets of A_1: $\{T\}$, $\{B\}$, $\{T, B\}$

Example

	L	R
T	$1,1$	$2,0$
B	$1,0$	$0,2$

- Subsets of A_1: $\{T\}$, $\{B\}$, $\{T,B\}$
- Subsets of A_2: $\{L\}$, $\{R\}$, $\{L,R\}$

Example

	L	R
T	$1,1$	$2,0$
B	$1,0$	$0,2$

- Subsets of A_1: $\{T\}$, $\{B\}$, $\{T,B\}$
- Subsets of A_2: $\{L\}$, $\{R\}$, $\{L,R\}$

Pairs of subsets:

P1	P2	NE?

Example

	L	R
T	$1, 1$	$2, 0$
B	$1, 0$	$0, 2$

- Subsets of A_1: $\{T\}$, $\{B\}$, $\{T, B\}$
- Subsets of A_2: $\{L\}$, $\{R\}$, $\{L, R\}$

Pairs of subsets:

P1	P2	NE?
$\{T\}$	$\{L\}$	

Example

	L	R
T	$1, 1$	$2, 0$
B	$1, 0$	$0, 2$

- Subsets of A_1: $\{T\}$, $\{B\}$, $\{T, B\}$
- Subsets of A_2: $\{L\}$, $\{R\}$, $\{L, R\}$

Pairs of subsets:

P1	P2	NE?
$\{T\}$	$\{L\}$	Yes

Example

	L	R
T	$1, 1$	$2, 0$
B	$1, 0$	$0, 2$

- Subsets of A_1: $\{T\}$, $\{B\}$, $\{T, B\}$
- Subsets of A_2: $\{L\}$, $\{R\}$, $\{L, R\}$

Pairs of subsets:

P1	P2	NE?
$\{T\}$	$\{L\}$	Yes
$\{T\}$	$\{R\}$	

Example

	L	R
T	$1, 1$	$2, 0$
B	$1, 0$	$0, 2$

- Subsets of A_1: $\{T\}$, $\{B\}$, $\{T, B\}$
- Subsets of A_2: $\{L\}$, $\{R\}$, $\{L, R\}$

Pairs of subsets:

P1	P2	NE?
$\{T\}$	$\{L\}$	Yes
$\{T\}$	$\{R\}$	No: R is not best response

Example

	L	R
T	$1, 1$	$2, 0$
B	$1, 0$	$0, 2$

- Subsets of A_1: $\{T\}$, $\{B\}$, $\{T, B\}$
- Subsets of A_2: $\{L\}$, $\{R\}$, $\{L, R\}$

Pairs of subsets:

P1	P2	NE?
$\{T\}$	$\{L\}$	Yes
$\{T\}$	$\{R\}$	No: R is not best response
$\{T\}$	$\{L, R\}$	

Example

	L	R
T	$1,1$	$2,0$
B	$1,0$	$0,2$

- Subsets of A_1: $\{T\}$, $\{B\}$, $\{T,B\}$
- Subsets of A_2: $\{L\}$, $\{R\}$, $\{L,R\}$

Pairs of subsets:

P1	P2	NE?
$\{T\}$	$\{L\}$	Yes
$\{T\}$	$\{R\}$	No: R is not best response
$\{T\}$	$\{L,R\}$	No: P2 not indifferent between L, R

Example

	L	R
T	$1, 1$	$2, 0$
B	$1, 0$	$0, 2$

- Subsets of A_1: $\{T\}$, $\{B\}$, $\{T, B\}$
- Subsets of A_2: $\{L\}$, $\{R\}$, $\{L, R\}$

Pairs of subsets:

P1	P2	NE?
$\{T\}$	$\{L\}$	Yes
$\{T\}$	$\{R\}$	No: R is not best response
$\{T\}$	$\{L, R\}$	No: P2 not indifferent between L, R
$\{B\}$	$\{L\}$	

Example

	L	R
T	$1,1$	$2,0$
B	$1,0$	$0,2$

- Subsets of A_1: $\{T\}$, $\{B\}$, $\{T,B\}$
- Subsets of A_2: $\{L\}$, $\{R\}$, $\{L,R\}$

Pairs of subsets:

P1	P2	NE?
$\{T\}$	$\{L\}$	Yes
$\{T\}$	$\{R\}$	No: R is not best response
$\{T\}$	$\{L,R\}$	No: P2 not indifferent between L, R
$\{B\}$	$\{L\}$	No: L is not best response

Example

	L	R
T	1, 1	2, 0
B	1, 0	0, 2

- Subsets of A_1: $\{T\}$, $\{B\}$, $\{T, B\}$
- Subsets of A_2: $\{L\}$, $\{R\}$, $\{L, R\}$

Pairs of subsets:

P1	P2	NE?
$\{T\}$	$\{L\}$	Yes
$\{T\}$	$\{R\}$	No: R is not best response
$\{T\}$	$\{L, R\}$	No: P2 not indifferent between L, R
$\{B\}$	$\{L\}$	No: L is not best response
$\{B\}$	$\{R\}$	

Example

	L	R
T	$1,1$	$2,0$
B	$1,0$	$0,2$

- Subsets of A_1: $\{T\}$, $\{B\}$, $\{T,B\}$
- Subsets of A_2: $\{L\}$, $\{R\}$, $\{L,R\}$

Pairs of subsets:

P1	P2	NE?
$\{T\}$	$\{L\}$	Yes
$\{T\}$	$\{R\}$	No: R is not best response
$\{T\}$	$\{L,R\}$	No: P2 not indifferent between L, R
$\{B\}$	$\{L\}$	No: L is not best response
$\{B\}$	$\{R\}$	No: B is not best response

Example

	L	R
T	$1, 1$	$2, 0$
B	$1, 0$	$0, 2$

- Subsets of A_1: $\{T\}$, $\{B\}$, $\{T, B\}$
- Subsets of A_2: $\{L\}$, $\{R\}$, $\{L, R\}$

Pairs of subsets:

P1	P2	NE?
$\{T\}$	$\{L\}$	Yes
$\{T\}$	$\{R\}$	No: R is not best response
$\{T\}$	$\{L, R\}$	No: P2 not indifferent between L, R
$\{B\}$	$\{L\}$	No: L is not best response
$\{B\}$	$\{R\}$	No: B is not best response
$\{B\}$	$\{L, R\}$	

Example

	L	R
T	$1, 1$	$2, 0$
B	$1, 0$	$0, 2$

- Subsets of A_1: $\{T\}$, $\{B\}$, $\{T, B\}$
- Subsets of A_2: $\{L\}$, $\{R\}$, $\{L, R\}$

Pairs of subsets:

P1	P2	NE?
$\{T\}$	$\{L\}$	Yes
$\{T\}$	$\{R\}$	No: R is not best response
$\{T\}$	$\{L, R\}$	No: P2 not indifferent between L, R
$\{B\}$	$\{L\}$	No: L is not best response
$\{B\}$	$\{R\}$	No: B is not best response
$\{B\}$	$\{L, R\}$	No: P2 not indifferent between L, R

Example

	L	R
T	$1,1$	$2,0$
B	$1,0$	$0,2$

- Subsets of A_1: $\{T\}$, $\{B\}$, $\{T,B\}$
- Subsets of A_2: $\{L\}$, $\{R\}$, $\{L,R\}$

Pairs of subsets:

P1	P2	NE?
$\{T\}$	$\{L\}$	Yes
$\{T\}$	$\{R\}$	No: R is not best response
$\{T\}$	$\{L,R\}$	No: P2 not indifferent between L, R
$\{B\}$	$\{L\}$	No: L is not best response
$\{B\}$	$\{R\}$	No: B is not best response
$\{B\}$	$\{L,R\}$	No: P2 not indifferent between L, R
$\{T,B\}$	$\{L\}$	

Example

	L	R
T	$1, 1$	$2, 0$
B	$1, 0$	$0, 2$

- Subsets of A_1: $\{T\}$, $\{B\}$, $\{T, B\}$
- Subsets of A_2: $\{L\}$, $\{R\}$, $\{L, R\}$

Pairs of subsets:

P1	P2	NE?
$\{T\}$	$\{L\}$	Yes
$\{T\}$	$\{R\}$	No: R is not best response
$\{T\}$	$\{L, R\}$	No: P2 not indifferent between L, R
$\{B\}$	$\{L\}$	No: L is not best response
$\{B\}$	$\{R\}$	No: B is not best response
$\{B\}$	$\{L, R\}$	No: P2 not indifferent between L, R
$\{T, B\}$	$\{L\}$	L is best response $\Leftrightarrow p \geq 2(1-p) \Leftrightarrow p \geq \frac{2}{3}$

Example

	L	R
T	$1, 1$	$2, 0$
B	$1, 0$	$0, 2$

- Subsets of A_1: $\{T\}$, $\{B\}$, $\{T, B\}$
- Subsets of A_2: $\{L\}$, $\{R\}$, $\{L, R\}$

Pairs of subsets:

P1	P2	NE?
$\{T\}$	$\{L\}$	Yes
$\{T\}$	$\{R\}$	No: R is not best response
$\{T\}$	$\{L, R\}$	No: P2 not indifferent between L, R
$\{B\}$	$\{L\}$	No: L is not best response
$\{B\}$	$\{R\}$	No: B is not best response
$\{B\}$	$\{L, R\}$	No: P2 not indifferent between L, R
$\{T, B\}$	$\{L\}$	L is best response $\Leftrightarrow p \geq 2(1-p) \Leftrightarrow p \geq \frac{2}{3}$
$\{T, B\}$	$\{R\}$	

Example

	L	R
T	$1,1$	$2,0$
B	$1,0$	$0,2$

- Subsets of A_1: $\{T\}$, $\{B\}$, $\{T,B\}$
- Subsets of A_2: $\{L\}$, $\{R\}$, $\{L,R\}$

Pairs of subsets:

P1	P2	NE?
$\{T\}$	$\{L\}$	Yes
$\{T\}$	$\{R\}$	No: R is not best response
$\{T\}$	$\{L,R\}$	No: P2 not indifferent between L, R
$\{B\}$	$\{L\}$	No: L is not best response
$\{B\}$	$\{R\}$	No: B is not best response
$\{B\}$	$\{L,R\}$	No: P2 not indifferent between L, R
$\{T,B\}$	$\{L\}$	L is best response $\Leftrightarrow p \geq 2(1-p) \Leftrightarrow p \geq \frac{2}{3}$
$\{T,B\}$	$\{R\}$	No: P1 not indifferent between T, B

Example

	L	R
T	$1, 1$	$2, 0$
B	$1, 0$	$0, 2$

- Subsets of A_1: $\{T\}$, $\{B\}$, $\{T, B\}$
- Subsets of A_2: $\{L\}$, $\{R\}$, $\{L, R\}$

Pairs of subsets:

P1	P2	NE?
$\{T\}$	$\{L\}$	Yes
$\{T\}$	$\{R\}$	No: R is not best response
$\{T\}$	$\{L, R\}$	No: P2 not indifferent between L, R
$\{B\}$	$\{L\}$	No: L is not best response
$\{B\}$	$\{R\}$	No: B is not best response
$\{B\}$	$\{L, R\}$	No: P2 not indifferent between L, R
$\{T, B\}$	$\{L\}$	L is best response $\Leftrightarrow p \geq 2(1-p) \Leftrightarrow p \geq \frac{2}{3}$
$\{T, B\}$	$\{R\}$	No: P1 not indifferent between T, B
$\{T, B\}$	$\{L, R\}$	

Example

	L	R
T	$1,1$	$2,0$
B	$1,0$	$0,2$

- Subsets of A_1: $\{T\}$, $\{B\}$, $\{T,B\}$
- Subsets of A_2: $\{L\}$, $\{R\}$, $\{L,R\}$

Pairs of subsets:

P1	P2	NE?
$\{T\}$	$\{L\}$	Yes
$\{T\}$	$\{R\}$	No: R is not best response
$\{T\}$	$\{L,R\}$	No: P2 not indifferent between L, R
$\{B\}$	$\{L\}$	No: L is not best response
$\{B\}$	$\{R\}$	No: B is not best response
$\{B\}$	$\{L,R\}$	No: P2 not indifferent between L, R
$\{T,B\}$	$\{L\}$	L is best response $\Leftrightarrow p \geq 2(1-p) \Leftrightarrow p \geq \frac{2}{3}$
$\{T,B\}$	$\{R\}$	No: P1 not indifferent between T, B
$\{T,B\}$	$\{L,R\}$	No: B not best response if $q < 1$

Example

	L	C	R
T	$1,2$	$3,3$	$1,1$
M	$1,0$	$0,2$	$2,1$
B	$3,4$	$2,1$	$0,7$

Example

	L	C	R
T	$1,2$	$3,3$	$1,1$
M	$1,0$	$0,2$	$2,1$
B	$3,4$	$2,1$	$0,7$

P1	P2	NE?
$\{T\}$	$\{L\}$	No: L is not BR

Example

	L	C	R
T	1, 2	3, 3	1, 1
M	1, 0	0, 2	2, 1
B	3, 4	2, 1	0, 7

P1	P2	NE?
$\{T\}$	$\{L\}$	No: L is not BR
$\{T\}$	$\{C\}$	Yes

Example

	L	C	R
T	$1,2$	$3,3$	$1,1$
M	$1,0$	$0,2$	$2,1$
B	$3,4$	$2,1$	$0,7$

P1	P2	NE?
$\{T\}$	$\{L\}$	No: L is not BR
$\{T\}$	$\{C\}$	Yes
Other singleton pairs		No: . . .

Example

	L	C	R
T	$1, 2$	$3, 3$	$1, 1$
M	$1, 0$	$0, 2$	$2, 1$
B	$3, 4$	$2, 1$	$0, 7$

P1	P2	NE?
$\{T\}$	$\{L\}$	No: L is not BR
$\{T\}$	$\{C\}$	Yes
Other singleton pairs		No: ...
$\{T\}$	$\{L, C\}$	No: P2 not indifferent between L, C

Example

	L	C	R
T	$1,2$	$3,3$	$1,1$
M	$1,0$	$0,2$	$2,1$
B	$3,4$	$2,1$	$0,7$

P1	P2	NE?
$\{T\}$	$\{L\}$	No: L is not BR
$\{T\}$	$\{C\}$	Yes
Other singleton pairs		No: ...
$\{T\}$	$\{L, C\}$	No: P2 not indifferent between L, C
Other pairs w/ 1 action in support for one player		Only one with indifference: $(\{T, M\}, \{L\})$. Not NE, because B is better than T, M.

Example

	L	C	R
T	$1, 2$	$3, 3$	$1, 1$
M	$1, 0$	$0, 2$	$2, 1$
B	$3, 4$	$2, 1$	$0, 7$

P1	P2	NE?
$\{T\}$	$\{L\}$	No: L is not BR
$\{T\}$	$\{C\}$	Yes
Other singleton pairs		No: ...
$\{T\}$	$\{L, C\}$	No: P2 not indifferent between L, C
Other pairs w/ 1 action in support for one player		Only one with indifference: $(\{T, M\}, \{L\})$. Not NE, because B is better than T, M.
$\{T, M\}$	$\{L, C\}$	No: L is not BR to any strategy with support $\{T, M\}$

Example

	L	C	R
T	$1, 2$	$3, 3$	$1, 1$
M	$1, 0$	$0, 2$	$2, 1$
B	$3, 4$	$2, 1$	$0, 7$

P1	P2	NE?
$\{T, B\}$	$\{L, C\}$	Find q s.t. P1 indifferent between T and B: $q = \frac{1}{3}$. Find p s.t. P2 indifferent between L and C: $p = \frac{3}{4}$. Now check payoffs to M and R: payoff to $M \leq$ payoff to T, B; payoff to $R \leq$ payoff to L, C $\Rightarrow$ Nash equilibrium.

Example

	L	C	R
T	$1, 2$	$3, 3$	$1, 1$
M	$1, 0$	$0, 2$	$2, 1$
B	$3, 4$	$2, 1$	$0, 7$

P1	P2	NE?
$\{T, B\}$	$\{L, C\}$	Find q s.t. P1 indifferent between T and B: $q = \frac{1}{3}$. Find p s.t. P2 indifferent between L and C: $p = \frac{3}{4}$. Now check payoffs to M and R: payoff to $M \leq$ payoff to T, B; payoff to $R \leq$ payoff to L, C $\Rightarrow$ Nash equilibrium.
Other pairs w/ 2 actions in each support		…

Example

	L	C	R
T	$1,2$	$3,3$	$1,1$
M	$1,0$	$0,2$	$2,1$
B	$3,4$	$2,1$	$0,7$

P1	P2	NE?
Pairs w/ 2 actions in support for 1 player, 3 actions in support for other		...

Example

	L	C	R
T	1, 2	3, 3	1, 1
M	1, 0	0, 2	2, 1
B	3, 4	2, 1	0, 7

P1	P2	NE?
Pairs w/ 2 actions in support for 1 player, 3 actions in support for other		...
$\{T, M, B\}$	$\{L, C, R\}$	For each player, three equations in three unknowns.

Finding all mixed strategy equilibria

- Method is exhaustive

Finding all mixed strategy equilibria

- Method is exhaustive
- ... and exhausting for even moderate sized games

Finding all mixed strategy equilibria

- Method is exhaustive
- . . . and exhausting for even moderate sized games
- Number of possible supports for mixed strategy of player with k actions:

Finding all mixed strategy equilibria

- Method is exhaustive
- . . . and exhausting for even moderate sized games
- Number of possible supports for mixed strategy of player with k actions: $2^k - 1$

Finding all mixed strategy equilibria

- Method is exhaustive
- ... and exhausting for even moderate sized games
- Number of possible supports for mixed strategy of player with k actions: $2^k - 1$
- So n players each with k actions $\Rightarrow (2^k - 1)^n$ possible pairs of supports

Finding all mixed strategy equilibria

- Method is exhaustive
- ... and exhausting for even moderate sized games
- Number of possible supports for mixed strategy of player with k actions: $2^k - 1$
- So n players each with k actions $\Rightarrow (2^k - 1)^n$ possible pairs of supports
- 4 players, 4 actions each $\Rightarrow (15)^4 \approx 50{,}000$ possible pairs

Finding all mixed strategy equilibria

- Method is exhaustive
- . . . and exhausting for even moderate sized games
- Number of possible supports for mixed strategy of player with k actions: $2^k - 1$
- So n players each with k actions $\Rightarrow (2^k - 1)^n$ possible pairs of supports
- 4 players, 4 actions each $\Rightarrow (15)^4 \approx 50{,}000$ possible pairs
- 2 players, 10 actions each $\Rightarrow (1{,}023)^2 \approx 1{,}000{,}000$ possible pairs

Finding all mixed strategy equilibria

- Method is exhaustive
- ... and exhausting for even moderate sized games
- Number of possible supports for mixed strategy of player with k actions: $2^k - 1$
- So n players each with k actions $\Rightarrow (2^k - 1)^n$ possible pairs of supports
- 4 players, 4 actions each $\Rightarrow (15)^4 \approx 50{,}000$ possible pairs
- 2 players, 10 actions each $\Rightarrow (1{,}023)^2 \approx 1{,}000{,}000$ possible pairs
- Computationally efficient methods exist to find *an* equilibrium

Finding all mixed strategy equilibria

- Method is exhaustive
- . . . and exhausting for even moderate sized games
- Number of possible supports for mixed strategy of player with k actions: $2^k - 1$
- So n players each with k actions $\Rightarrow (2^k - 1)^n$ possible pairs of supports
- 4 players, 4 actions each $\Rightarrow (15)^4 \approx 50{,}000$ possible pairs
- 2 players, 10 actions each $\Rightarrow (1{,}023)^2 \approx 1{,}000{,}000$ possible pairs
- Computationally efficient methods exist to find *an* equilibrium
- See `http://www.gambit-project.org/` and Chapter 4 of *Multiagent Systems* by Shoham and Leyton-Brown

Strict Nash equilibrium

Definition

A Nash equilibrium is strict if, for every player, the payoff to every nonequilibrium strategy is *less than* the payoff to her equilibrium strategy, given the other players' strategies

Strict Nash equilibrium

Definition

A Nash equilibrium is strict if, for every player, the payoff to every nonequilibrium strategy is *less than* the payoff to her equilibrium strategy, given the other players' strategies

Examples

- Nash equilibrium of *Prisoner's Dilemma*

Strict Nash equilibrium

Definition

A Nash equilibrium is strict if, for every player, the payoff to every nonequilibrium strategy is *less than* the payoff to her equilibrium strategy, given the other players' strategies

Examples

- Nash equilibrium of *Prisoner's Dilemma* is strict

Strict Nash equilibrium

Definition

A Nash equilibrium is strict if, for every player, the payoff to every nonequilibrium strategy is *less than* the payoff to her equilibrium strategy, given the other players' strategies

Examples

- Nash equilibrium of *Prisoner's Dilemma* is strict
- Pure strategy Nash equilibria of *BoS*

Strict Nash equilibrium

Definition

A Nash equilibrium is strict if, for every player, the payoff to every nonequilibrium strategy is *less than* the payoff to her equilibrium strategy, given the other players' strategies

Examples

- Nash equilibrium of *Prisoner's Dilemma* is strict
- Pure strategy Nash equilibria of *BoS* are strict

Strict Nash equilibrium

Definition

A Nash equilibrium is strict if, for every player, the payoff to every nonequilibrium strategy is *less than* the payoff to her equilibrium strategy, given the other players' strategies

Examples

- Nash equilibrium of *Prisoner's Dilemma* is strict
- Pure strategy Nash equilibria of *BoS* are strict
- Nash equilibrium of Bertrand's duopoly game

Strict Nash equilibrium

Definition

A Nash equilibrium is strict if, for every player, the payoff to every nonequilibrium strategy is *less than* the payoff to her equilibrium strategy, given the other players' strategies

Examples

- Nash equilibrium of *Prisoner's Dilemma* is strict
- Pure strategy Nash equilibria of *BoS* are strict
- Nash equilibrium of Bertrand's duopoly game is not strict

Strict Nash equilibrium

Definition

A Nash equilibrium is strict if, for every player, the payoff to every nonequilibrium strategy is *less than* the payoff to her equilibrium strategy, given the other players' strategies

Examples

- Nash equilibrium of *Prisoner's Dilemma* is strict
- Pure strategy Nash equilibria of *BoS* are strict
- Nash equilibrium of Bertrand's duopoly game is not strict
- Mixed strategy Nash equilibrium in which some player's strategy is not pure

Strict Nash equilibrium

Definition

A Nash equilibrium is strict if, for every player, the payoff to every nonequilibrium strategy is *less than* the payoff to her equilibrium strategy, given the other players' strategies

Examples

- Nash equilibrium of *Prisoner's Dilemma* is strict
- Pure strategy Nash equilibria of *BoS* are strict
- Nash equilibrium of Bertrand's duopoly game is not strict
- Mixed strategy Nash equilibrium in which some player's strategy is not pure is not strict (all actions in support of strategy yield same payoff)

Incentives in mixed strategy Nash equilibrium

- *Every* mixed strategy with the same support as equilibrium mixed strategy is best response to other players' strategies

Incentives in mixed strategy Nash equilibrium

- *Every* mixed strategy with the same support as equilibrium mixed strategy is best response to other players' strategies
- So no player has a positive incentive to choose equilibrium strategy

Incentives in mixed strategy Nash equilibrium

- *Every* mixed strategy with the same support as equilibrium mixed strategy is best response to other players' strategies
- So no player has a positive incentive to choose equilibrium strategy
- What determines her equilibrium strategy?

Incentives in mixed strategy Nash equilibrium

- *Every* mixed strategy with the same support as equilibrium mixed strategy is best response to other players' strategies
- So no player has a positive incentive to choose equilibrium strategy
- What determines her equilibrium strategy?
- Strategy is determined by requirement that *other* players' strategies be optimal

Incentives in mixed strategy Nash equilibrium

- *Every* mixed strategy with the same support as equilibrium mixed strategy is best response to other players' strategies
- So no player has a positive incentive to choose equilibrium strategy
- What determines her equilibrium strategy?
- Strategy is determined by requirement that *other* players' strategies be optimal
- Specifically, in *two-player* game, one player's equilibrium mixed strategy keeps *other* player indifferent between a set of her actions, so that *she* is willing to randomize

	$L\ (\frac{2}{3})$	$R\ (\frac{1}{3})$
$T\ (\frac{1}{5})$	$1, 0$	$0, 4$
$B\ (\frac{4}{5})$	$0, 1$	$2, 0$

Example: reporting a crime (“volunteer’s dilemma”)

- Many people witness a crime

Example: reporting a crime (“volunteer’s dilemma”)

- Many people witness a crime
- One person’s reporting crime to police suffices

Example: reporting a crime ("volunteer's dilemma")

- Many people witness a crime
- One person's reporting crime to police suffices
- When deciding whether to report, each person doesn't know whether anyone else has reported

Example: reporting a crime ("volunteer's dilemma")

- Many people witness a crime
- One person's reporting crime to police suffices
- When deciding whether to report, each person doesn't know whether anyone else has reported
- A person who reports bears a cost c

Example: reporting a crime ("volunteer's dilemma")

- Many people witness a crime
- One person's reporting crime to police suffices
- When deciding whether to report, each person doesn't know whether anyone else has reported
- A person who reports bears a cost c
- If the crime is reported, everyone obtains the benefit $v > c$

Application: reporting a crime ("volunteer's dilemma")

Strategic game

Players n individuals

Application: reporting a crime (“volunteer’s dilemma”)

Strategic game

Players n individuals

Actions For each player, $\{Call, Don't\ call\}$

Application: reporting a crime ("volunteer's dilemma")

Strategic game

Players n individuals

Actions For each player, $\{Call, Don't\ call\}$

Payoffs For each player i,

$$u_i(a) = \begin{cases} & \text{if } a_i = Call \\ & \text{if } a_i = Don't\ call \text{ and} \\ & \quad a_j = Call \text{ for some } j \neq i \\ & \text{if } a_j = Don't\ call \text{ for all } j \end{cases}$$

Application: reporting a crime ("volunteer's dilemma")

Strategic game

Players n individuals

Actions For each player, $\{Call, Don't\ call\}$

Payoffs For each player i,

$$u_i(a) = \begin{cases} v - c & \text{if } a_i = Call \\ & \text{if } a_i = Don't\ call \text{ and} \\ & \quad a_j = Call \text{ for some } j \neq i \\ & \text{if } a_j = Don't\ call \text{ for all } j \end{cases}$$

Application: reporting a crime ("volunteer's dilemma")

Strategic game

Players n individuals

Actions For each player, $\{Call, Don't\ call\}$

Payoffs For each player i,

$$u_i(a) = \begin{cases} v - c & \text{if } a_i = Call \\ v & \text{if } a_i = Don't\ call \text{ and} \\ & \quad a_j = Call \text{ for some } j \neq i \\ & \text{if } a_j = Don't\ call \text{ for all } j \end{cases}$$

Application: reporting a crime ("volunteer's dilemma")

Strategic game

Players n individuals

Actions For each player, $\{Call, Don't\ call\}$

Payoffs For each player i,

$$u_i(a) = \begin{cases} v - c & \text{if } a_i = Call \\ v & \text{if } a_i = Don't\ call \text{ and} \\ & \quad a_j = Call \text{ for some } j \neq i \\ 0 & \text{if } a_j = Don't\ call \text{ for all } j \end{cases}$$

Application: reporting a crime (“volunteer’s dilemma”)

Nash equilibria

- Equilibria in pure strategies?

Application: reporting a crime (“volunteer’s dilemma”)

Nash equilibria

- Equilibria in pure strategies? n pure Nash equilibria, in each of which exactly one player calls

Application: reporting a crime (“volunteer’s dilemma”)

Nash equilibria

- Equilibria in pure strategies? n pure Nash equilibria, in each of which exactly one player calls
- How can these equilibria be realized? For an equilibrium in which player 1 calls, who is player 1?

Application: reporting a crime ("volunteer's dilemma")

Nash equilibria

- Equilibria in pure strategies? n pure Nash equilibria, in each of which exactly one player calls
- How can these equilibria be realized? For an equilibrium in which player 1 calls, who is player 1?
- Look for symmetric equilibrium, in mixed strategies

Application: reporting a crime (“volunteer’s dilemma”)

Mixed strategy Nash equilibrium

In mixed strategy equilibrium in which every player calls with same probability p with $0 < p < 1$,

Application: reporting a crime (“volunteer’s dilemma”)

Mixed strategy Nash equilibrium

In mixed strategy equilibrium in which every player calls with same probability p with $0 < p < 1$,

$$\text{payoff if player calls} = \text{payoff if player doesn't call}$$

$\Rightarrow$

Application: reporting a crime (“volunteer’s dilemma”)

Mixed strategy Nash equilibrium

In mixed strategy equilibrium in which every player calls with same probability p with $0 < p < 1$,

$$\text{payoff if player calls} = \text{payoff if player doesn't call}$$

$\Rightarrow$

$v - c =$

Application: reporting a crime (“volunteer’s dilemma”)

Mixed strategy Nash equilibrium

In mixed strategy equilibrium in which every player calls with same probability p with $0 < p < 1$,

$$\text{payoff if player calls} = \text{payoff if player doesn't call}$$

$\Rightarrow$

$$v - c = 0 \cdot \Pr\{\text{no one else calls}\} + v \cdot \Pr\{\geq \text{one other person calls}\}$$

$\Rightarrow$

Application: reporting a crime (“volunteer’s dilemma”)

Mixed strategy Nash equilibrium

In mixed strategy equilibrium in which every player calls with same probability p with $0 < p < 1$,

$$\text{payoff if player calls} = \text{payoff if player doesn't call}$$

$\Rightarrow$

$$v - c = 0 \cdot \Pr\{\text{no one else calls}\} + v \cdot \Pr\{\geq \text{one other person calls}\}$$

$\Rightarrow$

$$v - c = v \cdot (1 - \Pr\{\text{no one else calls}\})$$

$\Rightarrow$

Application: reporting a crime (“volunteer’s dilemma”)

Mixed strategy Nash equilibrium

In mixed strategy equilibrium in which every player calls with same probability p with $0 < p < 1$,

$$\text{payoff if player calls} = \text{payoff if player doesn't call}$$

$\Rightarrow$

$$v - c = 0 \cdot \Pr\{\text{no one else calls}\} + v \cdot \Pr\{\geq \text{one other person calls}\}$$

$\Rightarrow$

$$v - c = v \cdot (1 - \Pr\{\text{no one else calls}\})$$

$\Rightarrow$

$$c/v = \Pr\{\text{No one else calls}\} =$$

$\Rightarrow$

Application: reporting a crime ("volunteer's dilemma")

Mixed strategy Nash equilibrium

In mixed strategy equilibrium in which every player calls with same probability p with $0 < p < 1$,

$$\text{payoff if player calls} = \text{payoff if player doesn't call}$$

$\Rightarrow$

$$v - c = 0 \cdot \Pr\{\text{no one else calls}\} + v \cdot \Pr\{\geq \text{one other person calls}\}$$

$\Rightarrow$

$$v - c = v \cdot (1 - \Pr\{\text{no one else calls}\})$$

$\Rightarrow$

$$c/v = \Pr\{\text{No one else calls}\} = (1 - p)^{n-1}$$

$\Rightarrow$

Application: reporting a crime ("volunteer's dilemma")

Mixed strategy Nash equilibrium

In mixed strategy equilibrium in which every player calls with same probability p with $0 < p < 1$,

$$\text{payoff if player calls} = \text{payoff if player doesn't call}$$

$\Rightarrow$

$$v - c = 0 \cdot \Pr\{\text{no one else calls}\} + v \cdot \Pr\{\geq \text{one other person calls}\}$$

$\Rightarrow$

$$v - c = v \cdot (1 - \Pr\{\text{no one else calls}\})$$

$\Rightarrow$

$$c/v = \Pr\{\text{No one else calls}\} = (1 - p)^{n-1}$$

$\Rightarrow$

$$p = 1 - (c/v)^{1/(n-1)}$$

Application: reporting a crime (“volunteer’s dilemma”)

Mixed strategy Nash equilibrium

In mixed strategy equilibrium in which every player calls with same probability p with $0 < p < 1$,

$$\text{payoff if player calls} = \text{payoff if player doesn't call}$$

$\Rightarrow$

$$v - c = 0 \cdot \Pr\{\text{no one else calls}\} + v \cdot \Pr\{\geq \text{one other person calls}\}$$

$\Rightarrow$

$$v - c = v \cdot (1 - \Pr\{\text{no one else calls}\})$$

$\Rightarrow$

$$c/v = \Pr\{\text{No one else calls}\} = (1-p)^{n-1}$$

$\Rightarrow$

$$p = 1 - (c/v)^{1/(n-1)}$$

Application: reporting a crime ("volunteer's dilemma")

Mixed strategy Nash equilibrium

- Conclusion: in a symmetric mixed strategy Nash equilibrium, every player calls with probability

$$p = 1 - (c/v)^{1/(n-1)}$$

(Note: this number is between 0 and 1.)

Application: reporting a crime ("volunteer's dilemma")

Mixed strategy Nash equilibrium: comparative statics

$$p = 1 - (c/v)^{1/(n-1)}$$

- $n \uparrow \Rightarrow p \downarrow$: more people $\Rightarrow$ each is less likely to call

Application: reporting a crime ("volunteer's dilemma")

Mixed strategy Nash equilibrium: comparative statics

$$p = 1 - (c/v)^{1/(n-1)}$$

- $n \uparrow \Rightarrow p \downarrow$: more people $\Rightarrow$ each is less likely to call
- Probability that at least one person calls:

$$\begin{aligned} &\Pr\{\text{at least one person calls}\} \\ &\qquad = 1 - \Pr\{\text{no one calls}\} \end{aligned}$$

Application: reporting a crime ("volunteer's dilemma")

Mixed strategy Nash equilibrium: comparative statics

$$p = 1 - (c/v)^{1/(n-1)}$$

- $n \uparrow \Rightarrow p \downarrow$: more people $\Rightarrow$ each is less likely to call
- Probability that at least one person calls:

$$\begin{aligned}
&\Pr\{\text{at least one person calls}\} \\
&\qquad = 1 - \Pr\{\text{no one calls}\} \\
&\qquad = 1 - \Pr\{i \text{ does not call}\} \Pr\{\text{no one else calls}\}
\end{aligned}$$

Application: reporting a crime ("volunteer's dilemma")

Mixed strategy Nash equilibrium: comparative statics

$$p = 1 - (c/v)^{1/(n-1)}$$

- $n \uparrow \Rightarrow p \downarrow$: more people $\Rightarrow$ each is less likely to call
- Probability that at least one person calls:

$$\begin{aligned}
\Pr\{\text{at least one person calls}\} \\
&= 1 - \Pr\{\text{no one calls}\} \\
&= 1 - \Pr\{i \text{ does not call}\} \Pr\{\text{no one else calls}\} \\
&= 1 -
\end{aligned}$$

Application: reporting a crime ("volunteer's dilemma")

Mixed strategy Nash equilibrium: comparative statics

$$p = 1 - (c/v)^{1/(n-1)}$$

- $n \uparrow \Rightarrow p \downarrow$: more people $\Rightarrow$ each is less likely to call
- Probability that at least one person calls:

$$\begin{aligned}
\Pr\{\text{at least one person calls}\} \\
&= 1 - \Pr\{\text{no one calls}\} \\
&= 1 - \Pr\{i \text{ does not call}\} \Pr\{\text{no one else calls}\} \\
&= 1 - (1 - p)
\end{aligned}$$

Application: reporting a crime ("volunteer's dilemma")

Mixed strategy Nash equilibrium: comparative statics

$$p = 1 - (c/v)^{1/(n-1)}$$

- $n \uparrow \Rightarrow p \downarrow$: more people $\Rightarrow$ each is less likely to call
- Probability that at least one person calls:

$$\begin{aligned}
&\Pr\{\text{at least one person calls}\} \\
&\qquad = 1 - \Pr\{\text{no one calls}\} \\
&\qquad = 1 - \Pr\{i \text{ does not call}\} \Pr\{\text{no one else calls}\} \\
&\qquad = 1 - (1-p)(c/v)
\end{aligned}$$

Application: reporting a crime ("volunteer's dilemma")

Mixed strategy Nash equilibrium: comparative statics

$$p = 1 - (c/v)^{1/(n-1)}$$

- $n \uparrow \Rightarrow p \downarrow$: more people $\Rightarrow$ each is less likely to call
- Probability that at least one person calls:

$$\begin{aligned}\Pr\{\text{at least one person calls}\} \\ &= 1 - \Pr\{\text{no one calls}\} \\ &= 1 - \Pr\{i \text{ does not call}\}\Pr\{\text{no one else calls}\} \\ &= 1 - (1-p)(c/v)\end{aligned}$$

Because $n \uparrow \Rightarrow p \downarrow$,

$$n \uparrow \Rightarrow \Pr\{\text{at least one person calls}\} \downarrow$$

Application: reporting a crime (“volunteer’s dilemma”)

Mixed strategy Nash equilibrium: comparative statics

$$p = 1 - (c/v)^{1/(n-1)}$$

- $n \uparrow \Rightarrow p \downarrow$: more people $\Rightarrow$ each is less likely to call
- Probability that at least one person calls:

$$\begin{aligned}\Pr\{\text{at least one person calls}\} \\ &= 1 - \Pr\{\text{no one calls}\} \\ &= 1 - \Pr\{i \text{ does not call}\} \Pr\{\text{no one else calls}\} \\ &= 1 - (1-p)(c/v)\end{aligned}$$

Because $n \uparrow \Rightarrow p \downarrow$,

$$n \uparrow \Rightarrow \Pr\{\text{at least one person calls}\} \downarrow$$

$\Rightarrow$ the more people, the *less* likely the police are informed

Domination by a mixed strategy

An action may be dominated by a *mixed strategy* even if it is not dominated by a pure strategy

Domination by a mixed strategy

An action may be dominated by a *mixed strategy* even if it is not dominated by a pure strategy

Example

	L	R
T	1	1
M	0	4
B	4	0

(where the payoffs are those of player 1)

Domination by a mixed strategy

An action may be dominated by a *mixed strategy* even if it is not dominated by a pure strategy

Example

	L	R
T	1	1
M	0	4
B	4	0

(where the payoffs are those of player 1)

- $\frac{1}{2} \cdot M \oplus \frac{1}{2} \cdot B$ strictly dominates T

Domination by a mixed strategy

An action may be dominated by a *mixed strategy* even if it is not dominated by a pure strategy

Example

	L	R
T	1	1
M	0	4
B	4	0

(where the payoffs are those of player 1)

- $\frac{1}{2} \cdot M \oplus \frac{1}{2} \cdot B$ strictly dominates T

Domination by a mixed strategy

An action may be dominated by a *mixed strategy* even if it is not dominated by a pure strategy

Example

	L	R
T	1	1
M	0	4
B	4	0

(where the payoffs are those of player 1)

- $\frac{1}{2} \cdot M \oplus \frac{1}{2} \cdot B$ strictly dominates T

An action strictly dominated by a mixed strategy is not used with positive probability in a mixed strategy equilibrium

Domination by a mixed strategy

An action may be dominated by a *mixed strategy* even if it is not dominated by a pure strategy

Example

	L	R
T	1	1
M	0	4
B	4	0

(where the payoffs are those of player 1)

- $\frac{1}{2} \cdot M \oplus \frac{1}{2} \cdot B$ strictly dominates T

An action strictly dominated by a mixed strategy is not used with positive probability in a mixed strategy equilibrium, and hence can be eliminated when looking for Nash equilibria

Mixed strategy equilibrium

- Interpretation: Read Section 3.2 of book

Mixed strategy equilibrium

- Interpretation: Read Section 3.2 of book
- Omit Section 3.3 (correlated equilibrium)

Mixed strategy equilibrium

- Interpretation: Read Section 3.2 of book
- Omit Section 3.3 (correlated equilibrium)
- Omit Section 3.4 (evolutionary equilibrium)

Another approach to outcomes in strategic games

Another approach to outcomes in strategic games

Definition

A belief of player i (about the other players' actions) is a probability distribution on $\times_{j \in N \setminus \{i\}} A_j$ (the set of lists of the other players' actions)

Another approach to outcomes in strategic games

Definition

A belief of player i (about the other players' actions) is a probability distribution on $\times_{j \in N \setminus \{i\}} A_j$ (the set of lists of the other players' actions)

Note: a belief may involve correlation between the other players' actions

Another approach to outcomes in strategic games

Definition

A belief of player i (about the other players' actions) is a probability distribution on $\times_{j \in N \setminus \{i\}} A_j$ (the set of lists of the other players' actions)

Note: a belief may involve correlation between the other players' actions

Definition

A player in a strategic game is rational if her mixed strategy is a best response to some belief

Implications of rationality

Every player

is rational

Implications of rationality

Every player

is rational

$\Updownarrow$

action is best response to a belief about other players' actions

Implications of rationality

Every player

is rational

$\Updownarrow$

action is best response to a belief about other players' actions

belief about other players' actions is correct

Implications of rationality

Every player

is rational

⇕

action is best response to a belief about other players' actions

belief about other players' actions is correct

↘ ↙

Nash equilibrium

Implications of rationality

Every player

is rational

⇕

action is best response to a belief about other players' actions

belief about other players' actions is correct

Implications of rationality

Which actions are best responses to some belief?

Implications of rationality

Which actions are best responses to some belief?

Proposition

An action is a best response to some belief if and only if it is not strictly dominated by a mixed strategy.

Implications of rationality

Which actions are best responses to some belief?

Proposition

An action is a best response to some belief if and only if it is not strictly dominated by a mixed strategy.

So every player is rational $\Rightarrow$ no player's strategy is strictly dominated by any mixed strategy

Implications of rationality

Which actions are best responses to some belief?

Proposition

An action is a best response to some belief if and only if it is not strictly dominated by a mixed strategy.

So every player is rational $\Rightarrow$ no player's strategy is strictly dominated by any mixed strategy

Example: Prisoner's Dilemma

		Player 2	
		Q	F
Player 1	Q	$3, 3$	$0, 4$
	F	$4, 0$	$1, 1$

Implications of rationality

Which actions are best responses to some belief?

Proposition

An action is a best response to some belief if and only if it is not strictly dominated by a mixed strategy.

So every player is rational $\Rightarrow$ no player's strategy is strictly dominated by any mixed strategy

Example: Prisoner's Dilemma

Player 2

		Q	F
Player 1	Q	$3, 3$	$0, 4$
	F	$4, 0$	$1, 1$

Players are rational $\Rightarrow$ action pair is (F, F)

Implications of rationality

Every player

is rational

$\Updownarrow$

no player's strategy is strictly dominated by a mixed strategy

belief about other players' actions is correct

$\searrow \quad \swarrow$

Nash equilibrium

Implications of rationality

Every player

is rational

$\Updownarrow$

no player's strategy is strictly dominated by a mixed strategy

belief about other players' actions is correct

Implications of rationality

Every player

is rational

⇕

no player's strategy is strictly dominated by a mixed strategy

. . . and believes that other players are rational

Implications of rationality

Every player

is rational

⇕

no player's strategy is strictly dominated by a mixed strategy

. . . and believes that other players are rational

. . . and believes that other players believe she is rational

Implications of rationality

Every player

is rational

⇕

no player's strategy is strictly dominated by a mixed strategy

. . . and believes that other players are rational

. . . and believes that other players believe she is rational

. . . and believes that other players believe she believes they are rational

Implications of rationality

Every player

is rational

$\Updownarrow$

no player's strategy is strictly dominated by a mixed strategy

. . . and believes that other players are rational

. . . and believes that other players believe she is rational

. . . and believes that other players believe she believes they are rational

. . . and so on

Implications of rationality

Every player

is rational

⇕

no player's strategy is strictly dominated by a mixed strategy

. . . and believes that other players are rational

. . . and believes that other players believe she is rational

. . . and believes that other players believe she believes they are rational

. . . and so on

?

Implications of rationality

Example

	L	C	R
T	$0, 4$	$4, 0$	$2, 1$
M	$1, 0$	$3, 1$	$3, 2$
B	$0, 2$	$2, 3$	$1, 1$

Implications of rationality

Example

	L	C	R
T	$0, 4$	$4, 0$	$2, 1$
M	$1, 0$	$3, 1$	$3, 2$
B	$0, 2$	$2, 3$	$1, 1$

- Player 1 is rational $\Rightarrow$

Implications of rationality

Example

	L	C	R
T	$0, 4$	$4, 0$	$2, 1$
M	$1, 0$	$3, 1$	$3, 2$
~~B~~	~~$0, 2$~~	~~$2, 3$~~	~~$1, 1$~~

- Player 1 is rational $\Rightarrow$ does not choose B (strictly dominated by M)

Implications of rationality

Example

	L	C	R
T	$0, 4$	$4, 0$	$2, 1$
M	$1, 0$	$3, 1$	$3, 2$
~~B~~	~~$0, 2$~~	~~$2, 3$~~	~~$1, 1$~~

- Player 1 is rational $\Rightarrow$ does not choose B (strictly dominated by M)
- Player 2 believes player 1 is rational $\Rightarrow$

Implications of rationality

Example

	L	C	R
T	$0, 4$	$4, 0$	$2, 1$
M	$1, 0$	$3, 1$	$3, 2$
~~B~~	~~$0, 2$~~	~~$2, 3$~~	~~$1, 1$~~

- Player 1 is rational $\Rightarrow$ does not choose B (strictly dominated by M)
- Player 2 believes player 1 is rational $\Rightarrow$ player 2 believes player 1 does not choose B

Implications of rationality

Example

	L	C	R
T	$0, 4$	$4, 0$	$2, 1$
M	$1, 0$	$3, 1$	$3, 2$
~~B~~	~~$0, 2$~~	~~$2, 3$~~	~~$1, 1$~~

- Player 1 is rational $\Rightarrow$ does not choose B (strictly dominated by M)
- Player 2 believes player 1 is rational $\Rightarrow$ player 2 believes player 1 does not choose B
  So player 2 is rational $\Rightarrow$

Implications of rationality

Example

	L	C	R
T	$0,4$	$4,0$	$2,1$
M	$1,0$	$3,1$	$3,2$
B	$0,2$	$2,3$	$1,1$

- Player 1 is rational $\Rightarrow$ does not choose B (strictly dominated by M)
- Player 2 believes player 1 is rational $\Rightarrow$ player 2 believes player 1 does not choose B
  So player 2 is rational $\Rightarrow$ she does not choose C (strictly dominated by R)

Implications of rationality

Example

	L	~~C~~	R
T	$0,4$	~~$4,0$~~	$2,1$
M	$1,0$	~~$3,1$~~	$3,2$
~~B~~	~~$0,2$~~	~~$2,3$~~	~~$1,1$~~

- Player 1 is rational $\Rightarrow$ does not choose B (strictly dominated by M)
- Player 2 believes player 1 is rational $\Rightarrow$ player 2 believes player 1 does not choose B
  So player 2 is rational $\Rightarrow$ she does not choose C (strictly dominated by R)
- Player 1 believes that player 2 is rational *and* that player 2 believes player 1 is rational $\Rightarrow$

Implications of rationality

Example

	L	C	R
T	$0, 4$	$4, 0$	$2, 1$
M	$1, 0$	$3, 1$	$3, 2$
B	$0, 2$	$2, 3$	$1, 1$

- Player 1 is rational $\Rightarrow$ does not choose B (strictly dominated by M)
- Player 2 believes player 1 is rational $\Rightarrow$ player 2 believes player 1 does not choose B
  So player 2 is rational $\Rightarrow$ she does not choose C (strictly dominated by R)
- Player 1 believes that player 2 is rational *and* that player 2 believes player 1 is rational $\Rightarrow$ player 1 believes player 2 believes player 1 does not choose B and that player 2 therefore does not choose C

Implications of rationality

Example

	L	C	R
T	$0,4$	$4,0$	$2,1$
M	$1,0$	$3,1$	$3,2$
B	$0,2$	$2,3$	$1,1$

- Player 1 is rational $\Rightarrow$ does not choose B (strictly dominated by M)
- Player 2 believes player 1 is rational $\Rightarrow$ player 2 believes player 1 does not choose B
  So player 2 is rational $\Rightarrow$ she does not choose C (strictly dominated by R)
- Player 1 believes that player 2 is rational *and* that player 2 believes player 1 is rational $\Rightarrow$ player 1 believes player 2 believes player 1 does not choose B and that player 2 therefore does not choose C
  So player 1 is rational $\Rightarrow$

Implications of rationality

Example

	L	C	R
~~T~~	~~0, 4~~	~~4, 0~~	~~2, 1~~
M	1, 0	~~3, 1~~	3, 2
~~B~~	~~0, 2~~	~~2, 3~~	~~1, 1~~

- Player 1 is rational $\Rightarrow$ does not choose B (strictly dominated by M)
- Player 2 believes player 1 is rational $\Rightarrow$ player 2 believes player 1 does not choose B
  So player 2 is rational $\Rightarrow$ she does not choose C (strictly dominated by R)
- Player 1 believes that player 2 is rational *and* that player 2 believes player 1 is rational $\Rightarrow$ player 1 believes player 2 believes player 1 does not choose B and that player 2 therefore does not choose C
  So player 1 is rational $\Rightarrow$ she does not choose T

Implications of rationality

Example

	L	*C*	*R*
T	0, 4	4, 0	2, 1
M	1, 0	3, 1	3, 2
B	0, 2	2, 3	1, 1

- Player 1 is rational $\Rightarrow$ does not choose *B* (strictly dominated by *M*)
- Player 2 believes player 1 is rational $\Rightarrow$ player 2 believes player 1 does not choose *B*
  So player 2 is rational $\Rightarrow$ she does not choose *C* (strictly dominated by *R*)
- Player 1 believes that player 2 is rational *and* that player 2 believes player 1 is rational $\Rightarrow$ player 1 believes player 2 believes player 1 does not choose *B* and that player 2 therefore does not choose *C*
  So player 1 is rational $\Rightarrow$ she does not choose *T*
- In one more step . . . player 2 does not choose *L*

Implications of rationality

Example

	~~L~~	~~C~~	R
~~T~~	~~0, 4~~	~~4, 0~~	~~2, 1~~
M	~~1, 0~~	~~3, 1~~	3, 2
~~B~~	~~0, 2~~	~~2, 3~~	~~1, 1~~

Conclusion

Every player is rational
and believes every other player is rational
and believes every other player believes she is rational
and so on . . .

$\Rightarrow$ only action pair that remains is (M, R)

Implications of rationality

- Each player's action in any action profile that survives iterated elimination of strictly dominated actions is rationalizable

Implications of rationality

- Each player's action in any action profile that survives iterated elimination of strictly dominated actions is rationalizable
  - Note: domination $=$ domination by a mixed strategy

Implications of rationality

- Each player's action in any action profile that survives iterated elimination of strictly dominated actions is rationalizable
  - Note: domination $=$ domination by a mixed strategy
- Any action used with positive probability in a mixed strategy Nash equilibrium is rationalizable

Implications of rationality

- Each player's action in any action profile that survives iterated elimination of strictly dominated actions is rationalizable
  - Note: domination $=$ domination by a mixed strategy
- Any action used with positive probability in a mixed strategy Nash equilibrium is rationalizable
- But in many games other actions also are rationalizable

Implications of rationality

- Each player's action in any action profile that survives iterated elimination of strictly dominated actions is rationalizable
  - Note: domination $=$ domination by a mixed strategy
- Any action used with positive probability in a mixed strategy Nash equilibrium is rationalizable
- But in many games other actions also are rationalizable
- If no action of any player is strictly dominated, then all actions of every player are rationalizable

Bayesian games

- Strategic game models situation in which each player knows preferences of other players

Bayesian games

- Strategic game models situation in which each player knows preferences of other players
- In some situations, players are not certain of other players' preferences

Bayesian games

- Strategic game models situation in which each player knows preferences of other players
- In some situations, players are not certain of other players' preferences
- Model of Bayesian Game allows players to face uncertainty about other players' preferences

Bayesian games: motivational example

Variant of *BoS* with imperfect information

- Player 1 doesn't know whether

Bayesian games: motivational example

Variant of *BoS* with imperfect information

- Player 1 doesn't know whether
  - player 2 prefers to go out with her—player 2 is type *m*

	B	*S*
B	2, 1	0, 0
S	0, 0	1, 2

meet

Bayesian games: motivational example

Variant of *BoS* with imperfect information

- Player 1 doesn't know whether
  - player 2 prefers to go out with her—player 2 is type m
  - or prefers to avoid her—player 2 is type v

	B	S
B	$2, 1$	$0, 0$
S	$0, 0$	$1, 2$

	B	S
B	$2, 0$	$0, 2$
S	$0, 1$	$1, 0$

meet ⟵ Two states ⟶ *avoid*

Bayesian games: motivational example

Variant of *BoS* with imperfect information

- Player 1 doesn't know whether
  - player 2 prefers to go out with her—player 2 is type m
  - or prefers to avoid her—player 2 is type v
- She thinks probabilities of states are $\frac{1}{2}$–$\frac{1}{2}$

1

	B	S
B	2, 1	0, 0
S	0, 0	1, 2

meet $(\frac{1}{2})$

	B	S
B	2, 0	0, 2
S	0, 1	1, 0

avoid $(\frac{1}{2})$

Bayesian games: motivational example

Variant of *BoS* with imperfect information

- Player 1 doesn't know whether
  - player 2 prefers to go out with her—player 2 is type *m*
  - or prefers to avoid her—player 2 is type *v*
- She thinks probabilities of states are $\frac{1}{2}$–$\frac{1}{2}$
- Player 2 knows player 1's preferences

1

2: *m*

	B	*S*
B	2, 1	0, 0
S	0, 0	1, 2

meet $(\frac{1}{2})$

2: *v*

	B	*S*
B	2, 0	0, 2
S	0, 1	1, 0

avoid $(\frac{1}{2})$

Bayesian games: motivational example

Variant of *BoS* with imperfect information

- Player 1 doesn't know whether
  - player 2 prefers to go out with her—player 2 is type m
  - or prefers to avoid her—player 2 is type v
- She thinks probabilities of states are $\frac{1}{2}$–$\frac{1}{2}$
- Player 2 knows player 1's preferences
- Probabilities are involved, so need players' preferences over lotteries, even if interested only in pure strategy equilibria $\Rightarrow$ Bernoulli payoffs

1

2: m

	B	S
B	$2, 1$	$0, 0$
S	$0, 0$	$1, 2$

meet $(\frac{1}{2})$

2: v

	B	S
B	$2, 0$	$0, 2$
S	$0, 1$	$1, 0$

avoid $(\frac{1}{2})$

Bayesian games: motivational example

Variant of *BoS* with imperfect information

1

2: *m*

	B	*S*
B	2, 1	0, 0
S	0, 0	1, 2

meet $(\frac{1}{2})$

2: *v*

	B	*S*
B	2, 0	0, 2
S	0, 1	1, 0

avoid $(\frac{1}{2})$

An equilibrium

Bayesian games: motivational example

Variant of *BoS* with imperfect information

1

2: *m*

	B	*S*
B	2, 1	0, 0
S	0, 0	1, 2

meet ($\frac{1}{2}$)

2: *v*

	B	*S*
B	2, 0	0, 2
S	0, 1	1, 0

avoid ($\frac{1}{2}$)

An equilibrium

- Player 1 chooses *B*

Bayesian games: motivational example

Variant of *BoS* with imperfect information

1

2: m

	B	S
B	$2, 1$	$0, 0$
S	$0, 0$	$1, 2$

meet $(\frac{1}{2})$

2: v

	B	S
B	$2, 0$	$0, 2$
S	$0, 1$	$1, 0$

avoid $(\frac{1}{2})$

An equilibrium

- Player 1 chooses B
- Type m of player 2 chooses B and type v chooses S

Bayesian games: motivational example

Variant of *BoS* with imperfect information

1

2: m

	B	S
B	$2, 1$	$0, 0$
S	$0, 0$	$1, 2$

meet $(\frac{1}{2})$

2: v

	B	S
B	$2, 0$	$0, 2$
S	$0, 1$	$1, 0$

avoid $(\frac{1}{2})$

An equilibrium

- Player 1 chooses B
- Type m of player 2 chooses B and type v chooses S
- Argument:

Bayesian games: motivational example

Variant of *BoS* with imperfect information

1

2: m

	B	S
B	$2, 1$	$0, 0$
S	$0, 0$	$1, 2$

meet $(\frac{1}{2})$

2: v

	B	S
B	$2, 0$	$0, 2$
S	$0, 1$	$1, 0$

avoid $(\frac{1}{2})$

An equilibrium

- Player 1 chooses B
- Type m of player 2 chooses B and type v chooses S
- Argument:
  - P1 chooses $B \Rightarrow$ payoff $\frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 0 = 1$; deviates to $S \Rightarrow$ payoff $\frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 1 = \frac{1}{2}$

Bayesian games: motivational example

Variant of *BoS* with imperfect information

1

2: m	B	S
B	$2, 1$	$0, 0$
S	$0, 0$	$1, 2$

meet $(\frac{1}{2})$

2: v	B	S
B	$2, 0$	$0, 2$
S	$0, 1$	$1, 0$

avoid $(\frac{1}{2})$

An equilibrium

- Player 1 chooses B
- Type m of player 2 chooses B and type v chooses S
- Argument:
  - P1 chooses $B \Rightarrow$ payoff $\frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 0 = 1$; deviates to $S \Rightarrow$ payoff $\frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 1 = \frac{1}{2}$
  - Type m of player 2: deviate to $S \Rightarrow$ payoff 0

Bayesian games: motivational example

Variant of *BoS* with imperfect information

1

2: m	B	S
B	$2, 1$	$0, 0$
S	$0, 0$	$1, 2$

meet $(\frac{1}{2})$

2: v	B	S
B	$2, 0$	$0, 2$
S	$0, 1$	$1, 0$

avoid $(\frac{1}{2})$

An equilibrium

- Player 1 chooses B
- Type m of player 2 chooses B and type v chooses S
- Argument:
  - P1 chooses $B \Rightarrow$ payoff $\frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 0 = 1$; deviates to $S \Rightarrow$ payoff $\frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 1 = \frac{1}{2}$
  - Type m of player 2: deviate to $S \Rightarrow$ payoff 0
  - Type v of player 2: deviate to $B \Rightarrow$ payoff 0

Bayesian games: motivational example

Another variant of *BoS* with imperfect information

Neither player knows whether other wants to go out with her

	B	*S*
B	2, 1	0, 0
S	0, 0	1, 2

State *mm*

Each player wants to go out with the other

Bayesian games: motivational example

Another variant of *BoS* with imperfect information

Neither player knows whether other wants to go out with her

	B	*S*
B	2, 1	0, 0
S	0, 0	1, 2

State *mm*

	B	*S*
B	2, 0	0, 2
S	0, 1	1, 0

State *mv*

1 wants to go out with 2, but 2 wants to avoid 1

Bayesian games: motivational example

Another variant of *BoS* with imperfect information

Neither player knows whether other wants to go out with her

	B	S
B	$2, 1$	$0, 0$
S	$0, 0$	$1, 2$

State *mm*

	B	S
B	$2, 0$	$0, 2$
S	$0, 1$	$1, 0$

State *mv*

	B	S
B	$0, 1$	$2, 0$
S	$1, 0$	$0, 2$

State *vm*

2 wants to go out with 1, but 1 wants to avoid 2

Bayesian games: motivational example

Another variant of *BoS* with imperfect information

Neither player knows whether other wants to go out with her

State *mm*

	B	S
B	2, 1	0, 0
S	0, 0	1, 2

State *mv*

	B	S
B	2, 0	0, 2
S	0, 1	1, 0

State *vm*

	B	S
B	0, 1	2, 0
S	1, 0	0, 2

State *vv*

	B	S
B	0, 0	2, 2
S	1, 1	0, 0

Neither player wants to go out with the other

Bayesian games: motivational example

Another variant of *BoS* with imperfect information

Neither player knows whether other wants to go out with her

	B	S
B	$2, 1$	$0, 0$
S	$0, 0$	$1, 2$

State *mm* ($\frac{1}{3}$)

	B	S
B	$2, 0$	$0, 2$
S	$0, 1$	$1, 0$

State *mv* ($\frac{1}{3}$)

	B	S
B	$0, 1$	$2, 0$
S	$1, 0$	$0, 2$

State *vm* ($\frac{1}{6}$)

	B	S
B	$0, 0$	$2, 2$
S	$1, 1$	$0, 0$

State *vv* ($\frac{1}{6}$)

Common prior beliefs over the states

Bayesian games: motivational example

Another variant of *BoS* with imperfect information

Neither player knows whether other wants to go out with her

1: m_1

	B	S
B	2, 1	0, 0
S	0, 0	1, 2

State *mm* ($\frac{1}{3}$)

	B	S
B	2, 0	0, 2
S	0, 1	1, 0

State *mv* ($\frac{1}{3}$)

1 can't distinguish states *mm* and *mv*

	B	S
B	0, 1	2, 0
S	1, 0	0, 2

State *vm* ($\frac{1}{6}$)

	B	S
B	0, 0	2, 2
S	1, 1	0, 0

State *vv* ($\frac{1}{6}$)

Bayesian games: motivational example

Another variant of *BoS* with imperfect information

Neither player knows whether other wants to go out with her

1: m_1

	B	S
B	2, 1	0, 0
S	0, 0	1, 2

State *mm* ($\frac{1}{3}$)

	B	S
B	2, 0	0, 2
S	0, 1	1, 0

State *mv* ($\frac{1}{3}$)

1: v_1

	B	S
B	0, 1	2, 0
S	1, 0	0, 2

State *vm* ($\frac{1}{6}$)

	B	S
B	0, 0	2, 2
S	1, 1	0, 0

State *vv* ($\frac{1}{6}$)

1 can't distinguish states *vm* and *vv*

Bayesian games: motivational example

Another variant of *BoS* with imperfect information

Neither player knows whether other wants to go out with her

$1: m_1$

	B	S
B	$2, 1$	$0, 0$
S	$0, 0$	$1, 2$

State mm $(\frac{1}{3})$

	B	S
B	$2, 0$	$0, 2$
S	$0, 1$	$1, 0$

State mv $(\frac{1}{3})$

$2: m_2$

$1: v_1$

	B	S
B	$0, 1$	$2, 0$
S	$1, 0$	$0, 2$

State vm $(\frac{1}{6})$

	B	S
B	$0, 0$	$2, 2$
S	$1, 1$	$0, 0$

State vv $(\frac{1}{6})$

2 can't distinguish states *mm* and *vm*

Bayesian games: motivational example

Another variant of *BoS* with imperfect information

Neither player knows whether other wants to go out with her

1: m_1

	B	S
B	2, 1	0, 0
S	0, 0	1, 2

State mm ($\frac{1}{3}$)

	B	S
B	2, 0	0, 2
S	0, 1	1, 0

State mv ($\frac{1}{3}$)

2: m_2

2: v_2

1: v_1

	B	S
B	0, 1	2, 0
S	1, 0	0, 2

State vm ($\frac{1}{6}$)

	B	S
B	0, 0	2, 2
S	1, 1	0, 0

State vv ($\frac{1}{6}$)

2 can't distinguish states mv and vv

Bayesian games: motivational example

Another variant of *BoS* with imperfect information

Neither player knows whether other wants to go out with her

1: m_1

	B	S
B	$2, 1$	$0, 0$
S	$0, 0$	$1, 2$

State mm $(\frac{1}{3})$

	B	S
B	$2, 0$	$0, 2$
S	$0, 1$	$1, 0$

State mv $(\frac{1}{3})$

2: m_2

2: v_2

1: v_1

	B	S
B	$0, 1$	$2, 0$
S	$1, 0$	$0, 2$

State vm $(\frac{1}{6})$

	B	S
B	$0, 0$	$2, 2$
S	$1, 1$	$0, 0$

State vv $(\frac{1}{6})$

Bayesian games: motivational example

Another variant of *BoS* with imperfect information

- How to model information?

Bayesian games: motivational example

Another variant of *BoS* with imperfect information

- How to model information?
- Each player receives signal about state before choosing action:

Bayesian games: motivational example

Another variant of *BoS* with imperfect information

- How to model information?
- Each player receives signal about state before choosing action:
  - Player 1 receives same signal, say m_1, in states *mm* and *mv*, and same signal, say $v_1 \neq m_1$, in states *vm* and *vv*

Bayesian games: motivational example

Another variant of *BoS* with imperfect information

- How to model information?
- Each player receives signal about state before choosing action:
  - Player 1 receives same signal, say m_1, in states *mm* and *mv*, and same signal, say $v_1 \neq m_1$, in states *vm* and *vv*
  - Player 2 receives same signal, say m_2, in states *mm* and *vm*, and same signal, say $v_2 \neq m_2$, in states *mv* and *vv*.

Bayesian games: motivational example

Another variant of *BoS* with imperfect information

- How to model information?
- Each player receives signal about state before choosing action:
  - Player 1 receives same signal, say m_1, in states *mm* and *mv*, and same signal, say $v_1 \neq m_1$, in states *vm* and *vv*
  - Player 2 receives same signal, say m_2, in states *mm* and *vm*, and same signal, say $v_2 \neq m_2$, in states *mv* and *vv*.
- Player i who receives signal t_i is type t_i of player i

Bayesian games: motivational example

Another variant of *BoS* with imperfect information

- How to model information?
- Each player receives signal about state before choosing action:
  - Player 1 receives same signal, say m_1, in states *mm* and *mv*, and same signal, say $v_1 \neq m_1$, in states *vm* and *vv*
  - Player 2 receives same signal, say m_2, in states *mm* and *vm*, and same signal, say $v_2 \neq m_2$, in states *mv* and *vv*.
- Player i who receives signal t_i is type t_i of player i
- Type m_1 of player 1's posterior belief: state is *mm* with probability $\frac{1}{2}$ and *mv* with probability $\frac{1}{2}$
  Type v_1 of player 1's posterior belief: state is *vm* with probability $\frac{1}{2}$ and *vv* with probability $\frac{1}{2}$

Bayesian games: motivational example

Another variant of *BoS* with imperfect information

- How to model information?
- Each player receives signal about state before choosing action:
  - Player 1 receives same signal, say m_1, in states mm and mv, and same signal, say $v_1 \neq m_1$, in states vm and vv
  - Player 2 receives same signal, say m_2, in states mm and vm, and same signal, say $v_2 \neq m_2$, in states mv and vv.
- Player i who receives signal t_i is type t_i of player i
- Type m_1 of player 1's posterior belief: state is mm with probability $\frac{1}{2}$ and mv with probability $\frac{1}{2}$
  Type v_1 of player 1's posterior belief: state is vm with probability $\frac{1}{2}$ and vv with probability $\frac{1}{2}$
- Type m_2 of player 2's posterior belief: state is mm with probability $\frac{2}{3}$ and vm with probability $\frac{1}{3}$
  Type v_2 of player 2's posterior belief: state is mv with probability $\frac{2}{3}$ and vv with probability $\frac{1}{3}$

Bayesian games

Elements new relative to strategic game are indicated in red

A Bayesian game consists of

Bayesian games

Elements new relative to strategic game are indicated in red

A Bayesian game consists of

- a finite set N (*players*)

Bayesian games

Elements new relative to strategic game are indicated in red

A Bayesian game consists of

- a finite set N (*players*)
- a set Ω (*states*)

Bayesian games

Elements new relative to strategic game are indicated in red

A Bayesian game consists of

- a finite set N (*players*)
- a set Ω (*states*)
- for each player $i \in N$

Bayesian games

Elements new relative to strategic game are indicated in red

A Bayesian game consists of

- a finite set N (*players*)
- a set Ω (*states*)
- for each player $i \in N$
  - a set A_i (*actions*)

Bayesian games

Elements new relative to strategic game are indicated in red

A Bayesian game consists of

- a finite set N (*players*)
- a set Ω (*states*)
- for each player $i \in N$
  - a set A_i (*actions*)
  - a set T_i (of *signals* that i may receive) and a function $\tau_i : \Omega \to T_i$ that associates a signal with each state (i's signal function)

Bayesian games

Elements new relative to strategic game are indicated in red

A Bayesian game consists of

- a finite set N (*players*)
- a set Ω (*states*)
- for each player $i \in N$
  - a set A_i (*actions*)
  - a set T_i (of *signals* that i may receive) and a function $\tau_i : \Omega \to T_i$ that associates a signal with each state (i's signal function)
  - a probability measure p_i on Ω (i's *prior belief*) with $p_i(\tau_i^{-1}(t_i)) > 0$ for all $t_i \in T_i$

Bayesian games

Elements new relative to strategic game are indicated in red

A Bayesian game consists of

- a finite set N (*players*)
- a set Ω (*states*)
- for each player $i \in N$
  - a set A_i (*actions*)
  - a set T_i (of *signals* that i may receive) and a function $\tau_i : \Omega \to T_i$ that associates a signal with each state (i's signal function)
  - a probability measure p_i on Ω (i's *prior belief*) with $p_i(\tau_i^{-1}(t_i)) > 0$ for all $t_i \in T_i$
  - a *preference relation* over probability distributions over $A \times \Omega$ (represented by the expected value of a Bernoulli payoff function).

Bayesian games

Elements new relative to strategic game are indicated in red

A Bayesian game consists of

- a finite set N (*players*)
- a set Ω (*states*)
- for each player $i \in N$
  - a set A_i (*actions*)
  - a set T_i (of *signals* that i may receive) and a function $\tau_i : \Omega \to T_i$ that associates a signal with each state (i's signal function)
  - a probability measure p_i on Ω (i's *prior belief*) with $p_i(\tau_i^{-1}(t_i)) > 0$ for all $t_i \in T_i$
  - a *preference relation* over probability distributions over $A \times \Omega$ (represented by the expected value of a Bernoulli payoff function).

Notes

- i has no information: $\tau_i(\omega) = \tau_i(\omega')$ for all ω, ω'

Bayesian games

Elements new relative to strategic game are indicated in red

A Bayesian game consists of

- a finite set N (*players*)
- a set Ω (*states*)
- for each player $i \in N$
  - a set A_i (*actions*)
  - a set T_i (of *signals* that i may receive) and a function $\tau_i : \Omega \to T_i$ that associates a signal with each state (i's signal function)
  - a probability measure p_i on Ω (i's *prior belief*) with $p_i(\tau_i^{-1}(t_i)) > 0$ for all $t_i \in T_i$
  - a *preference relation* over probability distributions over $A \times \Omega$ (represented by the expected value of a Bernoulli payoff function).

Notes

- i has no information: $\tau_i(\omega) = \tau_i(\omega')$ for all ω, ω'
- i has perfect information: $\tau_i(\omega) \neq \tau_i(\omega')$ if $\omega \neq \omega'$

First example

Players

States

Actions

Signals

Beliefs

Payoffs

First example

Players $N = \{1, 2\}$ (the pair of people)

States

Actions

Signals

Beliefs

Payoffs

First example

Players $N = \{1, 2\}$ (the pair of people)

States $\Omega = \{meet, avoid\}$

Actions

Signals

Beliefs

Payoffs

First example

Players $N = \{1, 2\}$ (the pair of people)

States $\Omega = \{meet, avoid\}$

Actions $A_1 = A_2 = \{B, S\}$

Signals

Beliefs

Payoffs

First example

Players $N = \{1, 2\}$ (the pair of people)

States $\Omega = \{meet, avoid\}$

Actions $A_1 = A_2 = \{B, S\}$

Signals $T_1 = \{z\}$ and $\tau_1(meet) = \tau_1(avoid) = z$
$T_2 = \{m, v\}$ and $\tau_2(meet) = m$ and $\tau_2(avoid) = v$

Beliefs

Payoffs

First example

Players $N = \{1, 2\}$ (the pair of people)

States $\Omega = \{meet, avoid\}$

Actions $A_1 = A_2 = \{B, S\}$

Signals $T_1 = \{z\}$ and $\tau_1(meet) = \tau_1(avoid) = z$
$T_2 = \{m, v\}$ and $\tau_2(meet) = m$ and $\tau_2(avoid) = v$

Beliefs $p_1(meet) = p_2(meet) = \frac{1}{2}$,
$p_1(avoid) = p_2(avoid) = \frac{1}{2}$

Payoffs

First example

Players $N = \{1, 2\}$ (the pair of people)

States $\Omega = \{meet, avoid\}$

Actions $A_1 = A_2 = \{B, S\}$

Signals $T_1 = \{z\}$ and $\tau_1(meet) = \tau_1(avoid) = z$
$T_2 = \{m, v\}$ and $\tau_2(meet) = m$ and $\tau_2(avoid) = v$

Beliefs $p_1(meet) = p_2(meet) = \frac{1}{2}$,
$p_1(avoid) = p_2(avoid) = \frac{1}{2}$

Payoffs The payoffs $u_i(a, meet)$ of each player i for all possible action pairs are given in the left panel of the figure on the earlier slide and the payoffs $u_i(a, avoid)$ are given in the right panel

Second example

Players

States

Actions

Signals

Beliefs

Payoffs

Second example

Players $N = \{1, 2\}$ (the pair of people)

States

Actions

Signals

Beliefs

Payoffs

Second example

Players	$N = \{1, 2\}$ (the pair of people)
States	$\Omega = \{mm, mv, vm, vv\}$
Actions	
Signals	
Beliefs	
Payoffs	

Second example

Players $N = \{1, 2\}$ (the pair of people)

States $\Omega = \{mm, mv, vm, vv\}$

Actions $A_1 = A_2 = \{B, S\}$

Signals

Beliefs

Payoffs

Second example

Players $N = \{1, 2\}$ (the pair of people)

States $\Omega = \{mm, mv, vm, vv\}$

Actions $A_1 = A_2 = \{B, S\}$

Signals $T_1 = \{m_1, v_1\}$, $\tau_1(mm) = \tau_1(mv) = m_1$, and $\tau_1(vm) = \tau_1(vv) = v_1$
$T_2 = \{m_2, v_2\}$, $\tau_2(mm) = \tau_2(vm) = m_2$, and $\tau_2(mv) = \tau_2(vv) = v_2$

Beliefs

Payoffs

Second example

Players $N = \{1, 2\}$ (the pair of people)

States $\Omega = \{mm, mv, vm, vv\}$

Actions $A_1 = A_2 = \{B, S\}$

Signals $T_1 = \{m_1, v_1\}$, $\tau_1(mm) = \tau_1(mv) = m_1$, and $\tau_1(vm) = \tau_1(vv) = v_1$
$T_2 = \{m_2, v_2\}$, $\tau_2(mm) = \tau_2(vm) = m_2$, and $\tau_2(mv) = \tau_2(vv) = v_2$

Beliefs $p_i(mm) = p_i(mv) = \frac{1}{3}$ and $p_i(vm) = p_i(vv) = \frac{1}{6}$ for $i = 1, 2$

Payoffs

Second example

Players $N = \{1, 2\}$ (the pair of people)

States $\Omega = \{mm, mv, vm, vv\}$

Actions $A_1 = A_2 = \{B, S\}$

Signals $T_1 = \{m_1, v_1\}$, $\tau_1(mm) = \tau_1(mv) = m_1$, and $\tau_1(vm) = \tau_1(vv) = v_1$
$T_2 = \{m_2, v_2\}$, $\tau_2(mm) = \tau_2(vm) = m_2$, and $\tau_2(mv) = \tau_2(vv) = v_2$

Beliefs $p_i(mm) = p_i(mv) = \frac{1}{3}$ and $p_i(vm) = p_i(vv) = \frac{1}{6}$ for $i = 1, 2$

Payoffs The payoffs $u_i(a, \omega)$ of each player i for all possible action pairs and states are given on the earlier slide

Second example: Nash equilibria

1: m_1

	B	S
B	$2, 1$	$0, 0$
S	$0, 0$	$1, 2$

State mm ($\frac{1}{3}$)

Posterior: $\frac{1}{2}$

	B	S
B	$2, 0$	$0, 2$
S	$0, 1$	$1, 0$

State mv ($\frac{1}{3}$)

Posterior: $\frac{1}{2}$

2: m_2

2: v_2

1: v_1

	B	S
B	$0, 1$	$2, 0$
S	$1, 0$	$0, 2$

State vm ($\frac{1}{6}$)

Posterior: $\frac{1}{2}$

	B	S
B	$0, 0$	$2, 2$
S	$1, 1$	$0, 0$

State vv ($\frac{1}{6}$)

Posterior: $\frac{1}{2}$

Payoffs

$$\tfrac{1}{2} \cdot 2 + \tfrac{1}{2} \cdot 0 = 1$$
$$\tfrac{1}{2} \cdot 0 + \tfrac{1}{2} \cdot 1 = \tfrac{1}{2}$$

$$\tfrac{1}{2} \cdot 0 + \tfrac{1}{2} \cdot 2 = 1$$
$$\tfrac{1}{2} \cdot 1 + \tfrac{1}{2} \cdot 0 = \tfrac{1}{2}$$

Nash equilibrium: $((B, B), (B, S))$ (analysis for player 1)

Second example: Nash equilibria

1: m_1

	B	S
B	2, 1	0, 0
S	0, 0	1, 2

State mm ($\frac{1}{3}$)

Posterior: $\frac{2}{3}$

	B	S
B	2, 0	0, 2
S	0, 1	1, 0

State mv ($\frac{1}{3}$)

Posterior: $\frac{2}{3}$

2: m_2

2: v_2

1: v_1

	B	S
B	0, 1	2, 0
S	1, 0	0, 2

State vm ($\frac{1}{6}$)

Posterior: $\frac{1}{3}$

	B	S
B	0, 0	2, 2
S	1, 1	0, 0

State vv ($\frac{1}{6}$)

Posterior: $\frac{1}{3}$

Payoffs: 1 0 0 2

Nash equilibrium: $((B, B), (B, S))$ (analysis for player 2)

Second example: Nash equilibria

1: m_1

	B	S
B	$2,1$	$0,0$
S	$0,0$	$1,2$

State mm $(\frac{1}{3})$

	B	S
B	$2,0$	$0,2$
S	$0,1$	$1,0$

State mv $(\frac{1}{3})$

2: m_2

2: v_2

1: v_1

	B	S
B	$0,1$	$2,0$
S	$1,0$	$0,2$

State vm $(\frac{1}{6})$

	B	S
B	$0,0$	$2,2$
S	$1,1$	$0,0$

State vv $(\frac{1}{6})$

Another Nash equilibrium: $((S,B),(S,S))$