ECO2030: Microeconomic Theory II, module 1

Lecture 2

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Existence of a Nash Equilibrium

Example: Matching Pennies

	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

- Game has no Nash equilibrium

Questions

- Under what conditions does a game have a Nash equilibrium?
- What can we expect the players to do in a game without a Nash equilibrium?

Existence of a Nash Equilibrium

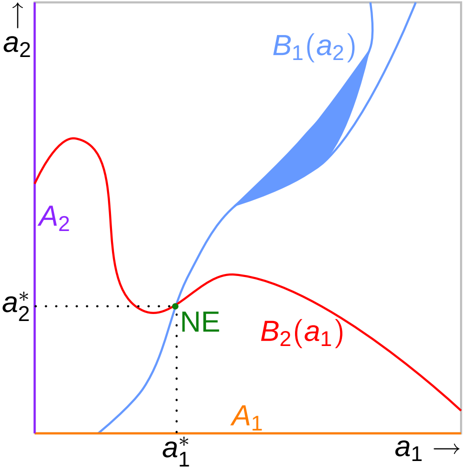
- Consider two-player game
- Nash equilibrium is action pair a^* such that

$$a_1^* \in B_1(a_2^*)$$
$$a_2^* \in B_2(a_1^*)$$

- Suppose each player's set of actions is set of real numbers

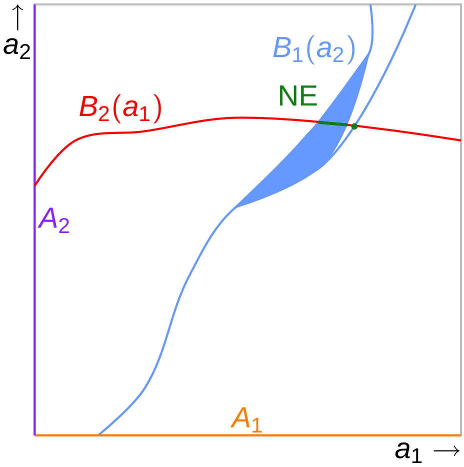
Existence of a Nash Equilibrium

Example



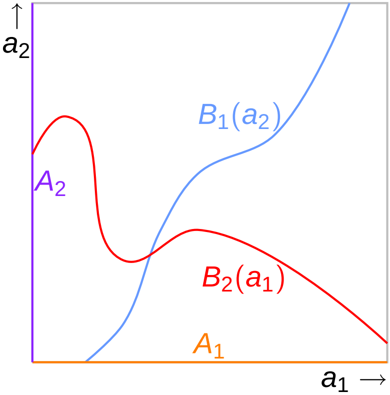
Existence of a Nash Equilibrium

Example



Existence of a Nash Equilibrium

What features of action sets and best response functions ensure that an equilibrium exists?



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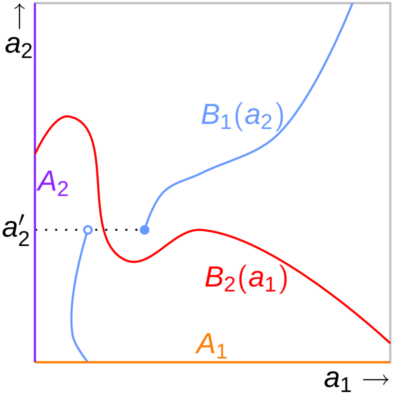
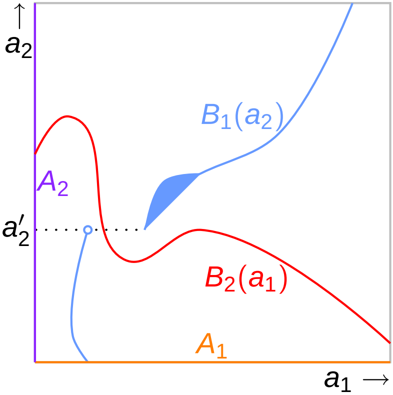


B_1 is not continuous at a'_2

- If best response functions are not continuous, game may have no Nash equilibrium

Existence of a Nash Equilibrium

What features of action sets and best response functions ensure that an equilibrium exists?

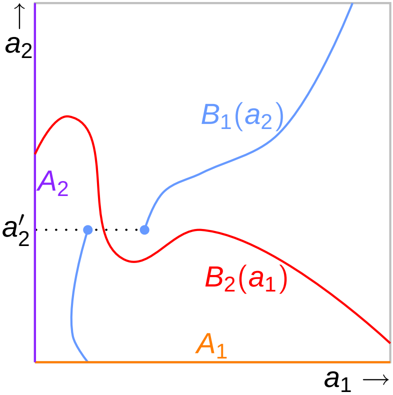


Graph of B_1 is not closed

- If best response functions do not have convex values, game may have no Nash equilibrium

Existence of a Nash Equilibrium

What features of action sets and best response functions ensure that an equilibrium exists?

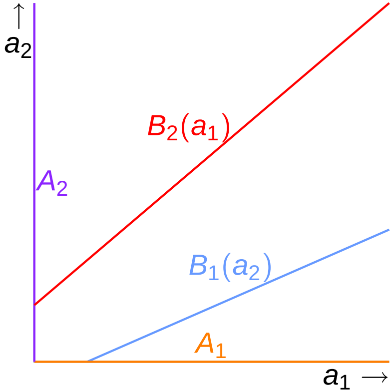


$B_1(a_2')$ is not convex

- If best response functions do not have closed graphs, game may have no Nash equilibrium

Existence of a Nash Equilibrium

What features of action sets and best response functions ensure that an equilibrium exists?

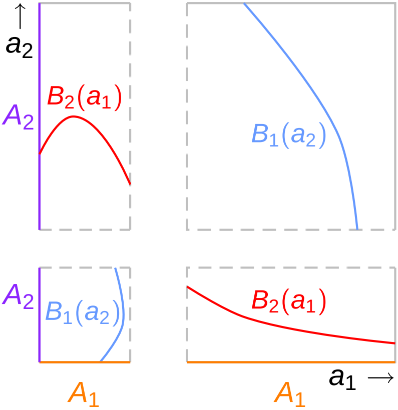


Action sets are not compact

- If action sets are not compact, game may have no Nash equilibrium

Existence of a Nash Equilibrium

What features of action sets and best response functions ensure that an equilibrium exists?



Action sets are not convex

- If action sets are not convex, game may have no Nash equilibrium

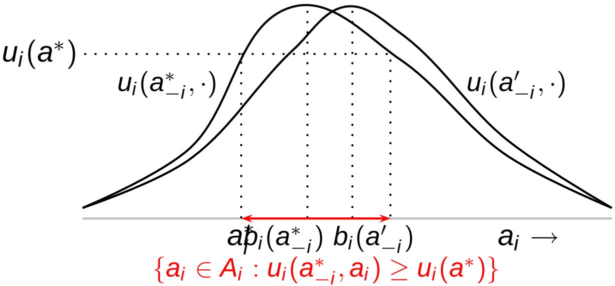
Existence of Nash Equilibrium

Sufficient conditions for existence of equilibrium

- action sets compact and convex
- best response functions convex-valued
- graphs of best response functions closed

Existence of Nash Equilibrium

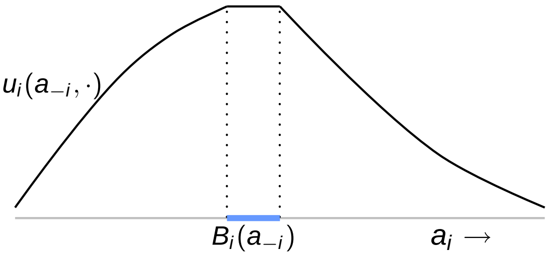
When are best response functions convex-valued, with closed graphs?



- u_i quasiconcave on A_i: $\{a_i \in A_i : u_i(a^*_{-i}, a_i) \geq u_i(a^*)\}$ is convex for every $a^* \in \times_{j \in N} A_j$
- u_i is continuous and quasiconcave on $A_i \Rightarrow$
  - b_i is continuous (B_i has a closed graph)
  - B_i is convex-valued

Existence of Nash Equilibrium

When are best response functions convex-valued, with closed graphs?



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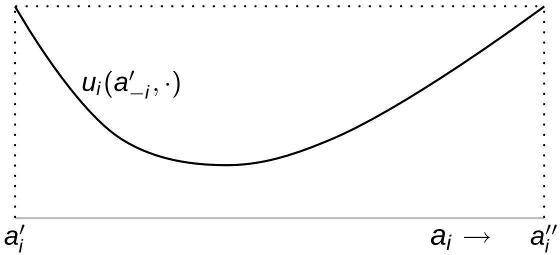
  - B_i is convex-valued

Existence of Nash Equilibrium

When are best response functions convex-valued, with closed graphs?



- u_i not quasiconcave $\Rightarrow B_i(a_{-i})$ may not be convex-valued:

$$B_i(a'_{-i}) = \{a'_i, a''_i\}$$

Existence of a Nash Equilibrium

Proposition

The strategic game $\langle N, (A_i), (\succsim_i)\rangle$ has a Nash equilibrium if for all $i \in N$

- the set A_i of actions of player i is a nonempty compact convex subset of a Euclidian space

and the preference relation $\succsim_i$ is

- continuous
- quasiconcave on A_i.

Notes

- Result gives *sufficient* conditions, not *necessary* ones: some games that do not satisfy the conditions have Nash equilibria
- Result says that a game has *at least* one Nash equilibrium

Existence of a Nash Equilibrium

- To prove result, need to show that there is profile a^* of actions such that

$$a_i^* \in B_i(a_{-i}^*) \text{ for all } i \in N$$

- Define set-valued function $B : \times_{j \in N} A_j \to \times_{j \in N} A_j$ by

$$B(a) = \times_{i \in N} B_i(a_{-i})$$

- Then the condition for equilibrium is

$$a^* \in B(a^*)$$

- Kakutani's fixed point theorem says that such an action profile a^* exists if $\times_{j \in N} A_j$ is compact and convex, $B(a)$ is nonempty and convex for all a, and the graph of B is closed

Existence of Nash equilibrium

Result in outline: NE exists if action sets compact and convex and payoff functions continuous and quasiconcave

Examples

- Games in which action sets are finite:
  - action sets not convex, so result does not apply
- Cournot's model of oligopoly:
  - action sets not bounded
  - if restrict actions to bounded sets, need payoff functions
    - continuous (e.g. no fixed costs)
    - quasiconcave (requires strong conditions on demand function and cost functions)
- Bertrand's model of oligopoly:
  - action sets not bounded
  - payoff functions not continuous

Games without Nash equilibria

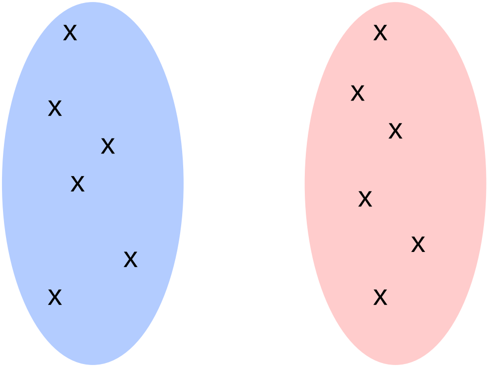
	L	R
T	$1, 0$	$0, 4$
B	$0, 1$	$2, 0$

What would happen if people played this game?

- Suppose large population of people who may play role of player 1 and large population of people who may play role of player 2
- In each of a series of periods, each member of population 1 is randomly matched with a member of population 2 to play the game
- What patterns of behavior could constitute a steady state?

Games without Nash equilibria: Steady state

	L	R
T	$1, 0$	$0, 4$
B	$0, 1$	$2, 0$



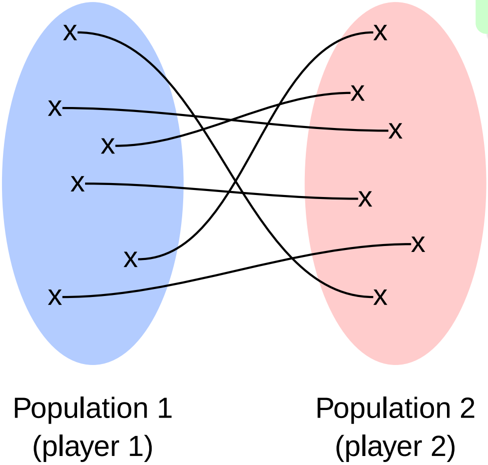
Population 1
(player 1)

Population 2
(player 2)

Games without Nash equilibria: Steady state

	L	*R*
T	1, 0	0, 4
B	0, 1	2, 0

Members of populations are randomly matched



Games without Nash equilibria: Steady state

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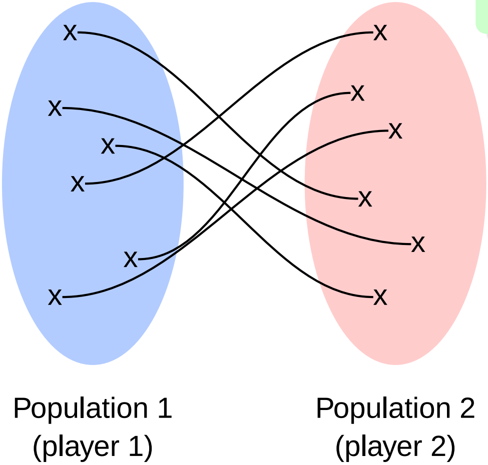
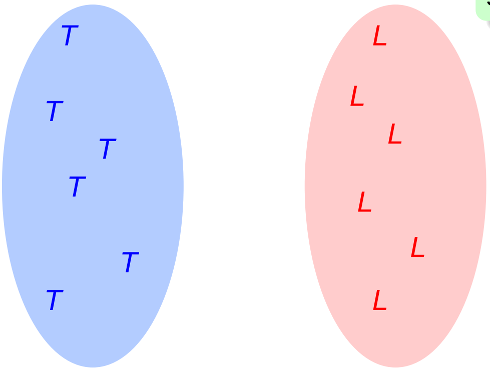


Games without Nash equilibria: Steady state

	L	R
T	$1, 0$	$0, 4$
B	$0, 1$	$2, 0$

Is this pattern of behavior a steady state?

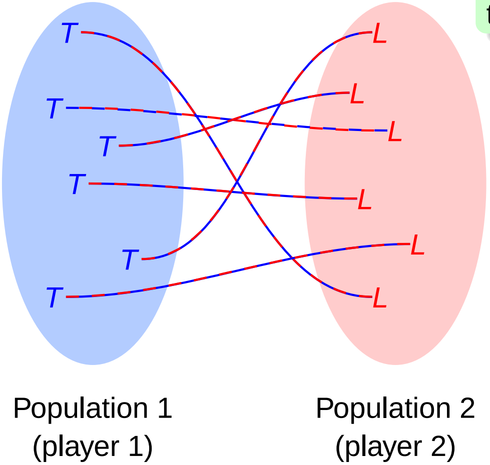


Population 1 (player 1)

Population 2 (player 2)

Games without Nash equilibria: Steady state

	L	R
T	$1, 0$	$0, 4$
B	$0, 1$	$2, 0$

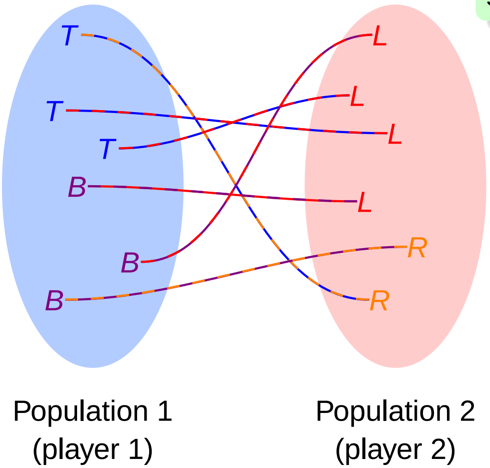


Not steady state: player 2's want to switch to R

Games without Nash equilibria: Steady state

	L	R
T	1, 0	0, 4
B	0, 1	2, 0

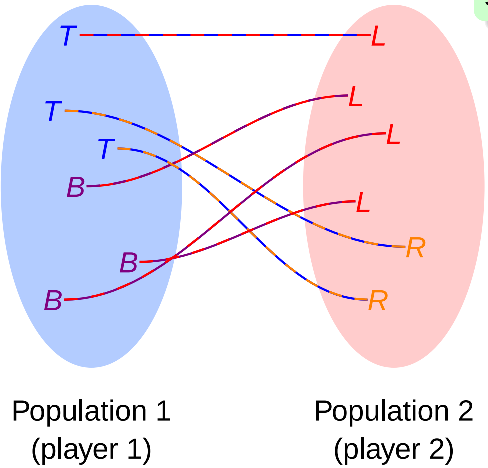
Is this pattern of behavior a steady state?



Games without Nash equilibria: Steady state

	L	R
T	$1, 0$	$0, 4$
B	$0, 1$	$2, 0$

Is this pattern of behavior a steady state?



Stochastic steady state

	L (q)	R $(1-q)$
T (p)	$1, 0$	$0, 4$
B $(1-p)$	$0, 1$	$2, 0$

Let

$$p = \text{fraction of population 1 choosing } T$$
$$q = \text{fraction of population 2 choosing } L$$

$$\begin{aligned} 0 < p < 1 &\Rightarrow \text{both } T \text{ and } B \text{ must be optimal for player 1} \\ &\Rightarrow \text{expected payoff to } T = \text{expected payoff to } B \\ &\Rightarrow q = 2(1-q) \\ &\Rightarrow q = \tfrac{2}{3} \end{aligned}$$

Stochastic steady state

	L (q)	R $(1-q)$
T (p)	$1, 0$	$0, 4$
B $(1-p)$	$0, 1$	$2, 0$

Similarly $0 < q < 1 \Rightarrow 1 - p = 4p$, so $p = \frac{1}{5}$

Thus game has stochastic steady state in which

- $\frac{1}{5}$ of population 1 chooses T, $\frac{4}{5}$ chooses B
- $\frac{2}{3}$ of population 2 chooses L, $\frac{1}{3}$ chooses R

And there is no other stochastic steady state. (What happens if all of population 1 chooses T? Or B? Or all of population 2 chooses L? Or R?)

Preferences and payoffs

What are we doing calculating expected values from payoffs that represent ordinal preferences?

- Players face uncertainty, so payoffs need to reflect preferences over lotteries
- Assume preferences satisfy vNM axioms $\Rightarrow$ represented by expected value of Bernoulli payoffs
- Each player i has a (Bernoulli) payoff function

$$u_i : \times_{j \in N} A_j \longrightarrow \mathbb{R}$$

  such that she evaluates a lottery over $\times_{j \in N} A_j$ by the expected value of u_i

Equivalent payoff representations

Example

	Q	F
Q	2, 2	0, 4
F	4, 0	1, 1

	Q	F
Q	2, 3	0, 5
F	5, 0	1, 1

- *Same* strategic game (numbers represent same *ordinal* preferences)
- If numbers interpreted as Bernoulli payoffs, expected values represent *different* preferences over lotteries:

Left game $(Q, Q) \sim_1 \frac{1}{2}(F, Q) \oplus \frac{1}{2}(Q, F)$

Right game $(Q, Q) \prec_1 \frac{1}{2}(F, Q) \oplus \frac{1}{2}(Q, F)$

Equivalent payoff representations

Expected values of payoff functions u and v represent same preferences over lotteries

$$\Longleftrightarrow$$

there exist numbers α and $\beta > 0$ such that

$$v(a) = \beta u(a) + \alpha \quad \text{for all } a \in \times_{j \in N} A_j$$

In words, Bernoulli payoffs are unique only up to *affine* transformations (not increasing transformations)

Strategic game with vNM preferences

Definition

A strategic game (with vNM preferences) consists of

- a finite set N (the set of *players*)
- for each player $i \in N$
  - a nonempty set A_i (the set of *actions* available to player i)
  - a function $u_i : \times_{j \in N} A_j \to \mathbb{R}$ whose expected value represents player i's preferences over the set of *lotteries* on $\times_{j \in N} A_j$

Two notions of steady state

1. Within each population, each player chooses an action, different players choosing different actions
2. Every player within each population chooses their action probabilistically, using the *same* probability distribution

- Same formal model captures both notions
- Subsequently mostly use language of second notion (it's easier)

Mixed extension of strategic game

The *mixed extension* of a strategic game expands the players' options to include randomizations

- $\Delta(A_i) =$ set of probability distributions over A_i
- Member of $\Delta(A_i)$: mixed strategy of player i
- Member of A_i: pure strategy of player i

Mixed extension of strategic game

Definition

The mixed extension of the strategic game $\langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$ is the following strategic game:

Players N

Action sets $\Delta(A_i)$ for player i

Preferences Represented by $U_i: \times_{j\in N} \Delta(A_j) \to \mathbb{R}$, where $U_i(\alpha)$ is the expected value under u_i of the lottery over $\times_{j\in N} A_j$ induced by α

If each A_j is finite then

$$U_i(\alpha) = \sum_{a\in\times_{j\in N}A_j} \left(\Pi_{j\in N}\alpha_j(a_j)\right) u_i(a)$$

Mixed strategy Nash equilibrium

Definition

A mixed strategy Nash equilibrium of a strategic game is a Nash equilibrium of the mixed extension of the game

Proposition (*Nash*)

Every strategic game in which each player has finitely many actions has a mixed strategy Nash equilibrium

Can prove this result by showing that mixed extension of a strategic game with finitely many actions satisfies conditions of earlier result about existence of a Nash equilibrium

How to find a mixed strategy Nash equilibrium?

- Mixed strategy Nash equilibrium $=$ Nash equilibrium of mixed extension
- So use techniques for finding Nash equilibrium

How to find mixed strategy Nash equilibria

Example

	L (q)	R $(1-q)$
T (p)	$1, 0$	$0, 4$
B $(1-p)$	$0, 1$	$2, 0$

Best responses of player 1:

- P1's expected payoff to T: $1 \cdot q + 0 \cdot (1-q) = q$
- P1's expected payoff to B: $0 \cdot q + 2 \cdot (1-q) = 2(1-q)$

$\Rightarrow$ $q < 2(1-q) \Rightarrow$ best response is $p = 0$ (i.e. B)
$q > 2(1-q) \Rightarrow$ best response is $p = 1$ (i.e. T)
$q = 2(1-q) \Rightarrow$ all mixed strategies are optimal

- $q = 2(1-q) \iff q = \frac{2}{3}$

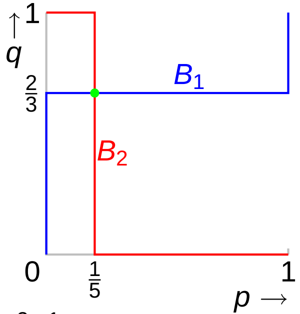
How to find mixed strategy Nash equilibria

Example

	L (q)	R ($1-q$)
T (p)	1, 0	0, 4
B ($1-p$)	0, 1	2, 0

Best response function of player 1:

$$B_1(q) = \begin{cases} \{0\} & \text{if } q < \frac{2}{3} \\ [0,1] & \text{if } q = \frac{2}{3} \\ \{1\} & \text{if } q > \frac{2}{3} \end{cases}$$



Unique Nash equilibrium: $((\frac{1}{5}, \frac{4}{5}), (\frac{2}{3}, \frac{1}{3}))$

Example: *BoS*

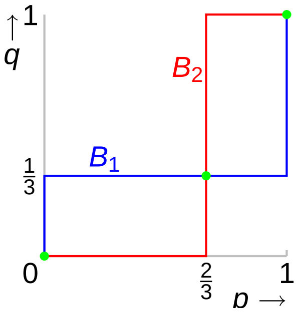
	B (q)	S ($1-q$)
B (p)	$2, 1$	$0, 0$
S ($1-p$)	$0, 0$	$1, 2$

Best responses of player 1

- P1's expected payoff to B: $2 \cdot q + 0 \cdot (1-q) = 2q$
- P1's expected payoff to S: $0 \cdot q + 1 \cdot (1-q) = 1-q$

$\Rightarrow$ P1's best response function:

$$B_1(q) = \begin{cases} \{0\} & \text{if } q < \frac{1}{3} \\ [0,1] & \text{if } q = \frac{1}{3} \\ \{1\} & \text{if } q > \frac{1}{3} \end{cases}$$



Three NEs: $((0,1),(0,1))$, $((\frac{2}{3},\frac{1}{3}),(\frac{1}{3},\frac{2}{3}))$, $((1,0),(1,0))$

Properties of mixed strategy Nash equilibrium

G: strategic game with ordinal preferences in which preferences of each player i are represented by payoff function u_i

G': strategic game with vNM preferences in which Bernoulli payoff function of each player i is u_i

Proposition

Any Nash equilibrium of G is a mixed strategy Nash equilibrium (in which each player's strategy is pure) of G'

- Player's payoff to mixed strategy is weighted average of payoffs to pure strategies to which mixed strategy assigns positive probability
- Hence mixed strategy may do *as well as* a pure strategy, but can never do *better than* all pure strategies
- Consequently pure strategy remains optimal when mixed strategies are allowed

Properties of mixed strategy Nash equilibrium

G: strategic game with ordinal preferences in which preferences of each player i are represented by payoff function u_i

G': strategic game with vNM preferences in which Bernoulli payoff function of each player i is u_i

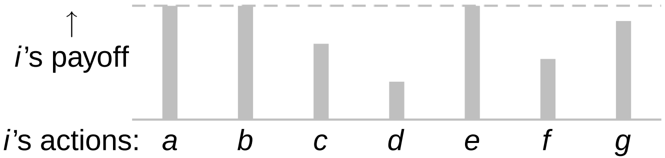
Proposition

Any mixed strategy Nash equilibrium of G' in which each player's strategy is pure is a Nash equilibrium of G

- If player optimally chooses a pure strategy when she is allowed to randomize, then when she is prohibited from randomizing the pure strategy remains optimal

Characterization of mixed strategy Nash equilibrium

- α^* is a mixed strategy Nash equilibrium $\Leftrightarrow \alpha_i^*$ is a best response to α_{-i}^* for all i
- When is a mixed strategy α_i a best response to α_{-i}^*?
- Suppose expected payoffs to player i's actions, given α_{-i}^*, are:



- What mixed strategies of player i are best responses to α_{-i}^*?
- Mixed strategy α_i is a best response to α_{-i}^* if and only if it assigns probability zero to c, d, and f; all probability must be assigned to actions that are best responses to α_{-i}^*

Characterization of mixed strategy Nash equilibrium

Definition

Support of mixed strategy $=$ set of actions to which strategy assigns positive probability

Proposition (*Lemma 33.2*)

α^* is a mixed strategy Nash equilibrium

$\Leftrightarrow$

for every player *i*, α_i^* is a best response to α_{-i}^*

$\Leftrightarrow$

for every player *i*, every action in support of α_i^* is a best response to α_{-i}^*.

Example

Is strategy pair a mixed strategy Nash equilibrium?

	L (0)	C ($\frac{1}{3}$)	R ($\frac{2}{3}$)	
T ($\frac{3}{4}$)	$\cdot, 2$	$3, 3$	$1, 1$	$\frac{5}{3}$
M (0)	$\cdot, \cdot$	$0, \cdot$	$2, \cdot$	$\frac{4}{3}$
B ($\frac{1}{4}$)	$\cdot, 4$	$5, 1$	$0, 7$	$\frac{5}{3}$
	$\frac{5}{2}$	$\frac{5}{2}$	$\frac{5}{2}$	

(Unspecified payoffs are irrelevant.)

- Compute expected payoff of each action, given other player's actions
- If every action in support of each player's mixed strategy yields same payoff *and* actions outside support yield at most this payoff *then* strategy pair is mixed strategy Nash equilibrium