ECO2030: Microeconomic Theory II, module 1

Lecture 1

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Single person decision problem

Model

A decision problem consists of

- a set A (the set of *actions*)
- a *preference relation* $\succsim$ on A

Theory

Decision-maker chooses $a^* \in A$ that is best according to $\succsim$:

$$a^* \succsim a \text{ for all } a \in A$$

Many decision-makers: Strategic games

Model

A strategic game consists of

- a finite set N (the set of *players*)
- for each player $i \in N$
  - a nonempty set A_i (the set of *actions* available to player i)
  - a *preference relation* $\succsim_i$ on $\times_{j \in N} A_j$.

Example

- $N = \{1, 2\}$
- $A_1 = A_2 = \{Q, F\}$
- $(F, Q) \succ_1 (Q, Q) \succ_1 (F, F) \succ_1 (Q, F)$
  $(Q, F) \succ_2 (Q, Q) \succ_2 (F, F) \succ_2 (F, Q)$

		Player 2	
		Q	F
Player 1	Q	$5, 10$	$1, 11$
	F	$8, -2$	$4, 1$

- Payoff representation isn't unique; any increasing function may be applied separately to each player's payoffs
- Story? Prisoner's Dilemma

Example: BoS

	Bach	*Stravinsky*
Bach	2, 1	0, 0
Stravinsky	0, 0	1, 2

Bach or Stravinsky?

Story

- Two people wish to go out together
- The options are concerts of music by Bach and by Stravinsky
- They want to go out together, but one prefers Bach and the other prefers Stravinsky
- If they go to different concerts, each of them is equally unhappy listening to the music of either composer

Example: Cournot's oligopoly game

Players $N = \{1, \ldots, n\}$ (firms)

Actions $A_i = [0, \infty)$ for $i = 1, \ldots, n$ (set of possible outputs)

Preferences Preferences of each firm are represented by payoff function u_i with

$$u_i(q_1, \ldots, q_n) = q_i P\left(\sum_{j=1}^{n} q_j\right) - C_i(q_i)$$

(firm i's profit), where $P : \mathbb{R}_+ \to \mathbb{R}_+$ ("inverse demand function") and $C_i : \mathbb{R}_+ \to \mathbb{R}_+$ (firm i's cost function).

Equilibrium

Every player

is rational

$\Updownarrow$

action is best given some belief about other players' actions

belief about other players' actions is correct

Every player's action is best given other player's actions

$\Rightarrow$ Nash equilibrium

Equilibrium

Every player

has played game many times previously, against variety of other players, and knows from her experience what other players will do

is rational

⇕

action is best given some belief about other players' actions

belief about other players' actions is correct

Every player's action is best given other player's actions

$\Rightarrow$ Nash equilibrium

Nash equilibrium

a^* is a *Nash equilibrium* if for all $i \in N$

$$a_i^* \text{ is optimal for } i \text{ according to } \succsim_i \text{ given } a_{-i}^*$$

Definition

A Nash equilibrium of a strategic game $\langle N, (A_i), (\succsim_i) \rangle$ is an action profile $a^* \in \times_{i \in N} A_i$ such that for all $i \in N$

$$(a_{-i}^*, a_i^*) \succsim_i (a_{-i}^*, a_i) \text{ for all } a_i \in A_i.$$

Prisoner's Dilemma

Player 2

		Q	F
Player 1	Q	$3, 3$	$0, 4$
	F	$4, 0$	$1, 1$

Check each action pair in turn:

- (Q, Q): not Nash equilibrium because if player 2 chooses Q, player 1 is better off choosing F than choosing Q
- (Q, F): not Nash equilibrium because ...
- (F, Q): not Nash equilibrium because ...
- (F, F): Nash equilibrium because F is at least as good as Q for each player if the other player chooses F

So: unique Nash equilibrium, (F, F)

BoS

	Bach	*Stravinsky*
Bach	$2, 1$	$0, 0$
Stravinsky	$0, 0$	$1, 2$

- Two Nash equilibria, $(Bach, Bach)$ and $(Stravinsky, Stravinsky)$
- Note: equilibria are not Pareto ranked

Matching Pennies

	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

- No Nash equilibrium!

Example

	L	R
T	$1, 1$	$2, 1$
B	$0, 0$	$2, 4$

- (T, L): Nash equilibrium
- (T, R): Nash equilibrium
- (B, L): Not Nash equilibrium
- (B, R): Nash equilibrium

Example: Cournot's model of oligopoly

Players $N = \{1, \ldots, n\}$ (firms).

Actions $A_i = [0, \infty)$ for $i = 1, \ldots, n$ (set of possible outputs).

Preferences Firm i's preferences are represented by payoff function u_i with

$$u_i(q_1, \ldots, q_n) = q_i P\left(\sum_{j=1}^{n} q_j\right) - C_i(q_i)$$

(i's profit), where P is an inverse demand function and C_i is firm i's cost function.

Can't examine every action pair in turn . . . Need a different technique

Best response functions

$$
\begin{aligned}
B_i(a_{-i}) &= \text{set of player } i\text{'s best actions given } a_{-i} \\
&= \{a_i \in A_i : (a_{-i}, a_i) \succsim_i (a_{-i}, a_i') \text{ for all } a_i' \in A_i\}
\end{aligned}
$$

In terms of payoffs,

$$B_i(a_{-i}) = \arg\max_{a_i} u_i(a_{-i}, a_i)$$

Nash equilibrium

$a^* \in \times_{i \in N} A_i$ is a Nash equilibrium if and only if

$$a_i^* \in B_i(a_{-i}^*) \text{ for all } i \in N$$

Best response functions

Procedure for finding Nash equilibria

1. Find best response function of each player
   - Optimization problem
2. Find all profiles a^* of actions for which

$$a_i^* \in B_i(a_{-i}^*) \text{ for all } i \in N$$

   - Set of conditions to be satisfied simultaneously

Games in which players have unique best responses

- Suppose each player i has *unique* best response to each a_{-i}:

$$B_i(a_{-i}) \text{ is a singleton for all } i \in N \text{ and all } a_{-i}$$

- Let $B_i(a_{-i}) = \{b_i(a_{-i})\}$ for all i and all a_{-i}
- Then

$$a^* \in A \text{ is Nash equilibrium } \Leftrightarrow a_i^* = b_i(a_{-i}^*) \text{ for all } i \in N$$

- Thus if set of players is $N = \{1, \ldots, n\}$, procedure is:
  1. find best response function b_i of each player i
  2. find solutions of set of n simultaneous equations

$$a_i^* = b_i(a_{-i}^*) \text{ for } i = 1, \ldots, n$$

  in n unknowns $a_1^*, \ldots, a_n^*$
- Problem 1 on Problem Set 1 asks you to use procedure to find Nash equilibria of example of Cournot's model

A less well-defined method of finding Nash equilibria

- Calculating complete best response function of every player is difficult in some games
- . . . and may not be necessary

Procedure for finding Nash equilibria

1. Explore players' best responses and isolate action profiles that appear to be equilibria
2. Prove that every such action profile is an equilibrium
3. Prove that no other action profile is an equilibrium

Example: Bertrand's model of oligopoly

Players $N = \{1, \ldots, n\}$ (firms)

Actions $A_i = [0, \infty)$ for $i = 1, \ldots, n$ (set of possible prices)

Preferences Firm i's preferences are represented by its profit:

$$u_i(p_1, \ldots, p_n) = \begin{cases} p_i \dfrac{D(p_i)}{m(p)} - C_i\left(\dfrac{D(p_i)}{m(p)}\right) & \text{if } p_i = \min_{j \in N} p_j \\ 0 & \text{if } p_i > \min_{j \in N} p_j \end{cases}$$

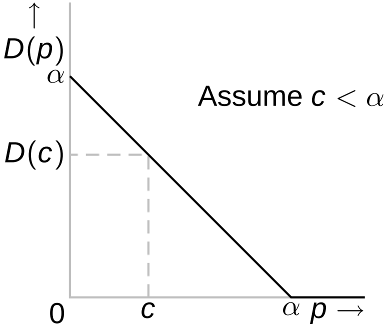
where

- D is demand function
- C_i is firm i's cost function with $C_i(0) = 0$
- $m(p)$ is number of firms j for which $p_j = \min_{k \in N} p_k$

Example of Bertrand's duopoly: constant unit cost and linear demand function

- Two firms: $n = 2$
- $C_i(q_i) = cq_i$ for $i = 1, 2$, and $c > 0$

$$D(p) = \begin{cases} \alpha - p & \text{if } p \le \alpha \\ 0 & \text{if } p > \alpha \end{cases}$$

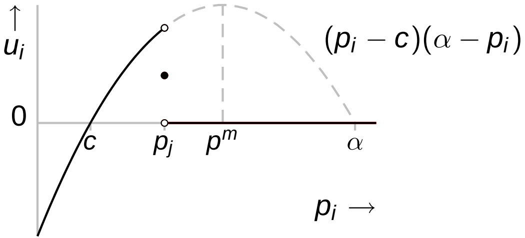


Example of Bertrand's duopoly

Assumptions $\Rightarrow$ payoff function of each firm i is

$$u_i(p_1, p_2) = \begin{cases} (p_i - c)(\alpha - p_i) & \text{if } p_i < p_j \\ \frac{1}{2}(p_i - c)(\alpha - p_i) & \text{if } p_i = p_j \\ 0 & \text{if } p_i > p_j \end{cases}$$

where j is the other firm ($j = 2$ if $i = 1$, and $j = 1$ if $i = 2$)



$c < p_j \leq p^m$

Example of Bertrand's duopoly

Assumptions ⇒ payoff function of each firm i is

$$u_i(p_1, p_2) = \begin{cases} (p_i - c)(\alpha - p_i) & \text{if } p_i < p_j \\ \frac{1}{2}(p_i - c)(\alpha - p_i) & \text{if } p_i = p_j \\ 0 & \text{if } p_i > p_j \end{cases}$$

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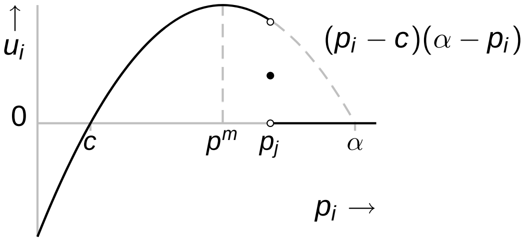
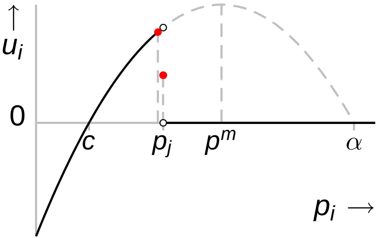


$p^m < p_j < \alpha$

Example of Bertrand's duopoly

Exploration

- $p_j > c \Rightarrow$ firm i gets almost twice as much profit by charging $p_j - \varepsilon$ than by charging p_j, for ε small
- $\Rightarrow$ strategic pressure to reduce prices?
- But prices less than c yield losses, so prices won't go below c
- Conclusion: (c, c) may be only equilibrium?



$c < p_j \leq p^m$

Example of Bertrand's duopoly

Proof that (c, c) is a Nash equilibrium

$$u_i(p_1, p_2) = \begin{cases} (p_i - c)(\alpha - p_i) & \text{if } p_i < p_j \\ \frac{1}{2}(p_i - c)(\alpha - p_i) & \text{if } p_i = p_j \\ 0 & \text{if } p_i > p_j \end{cases}$$

- $u_1(c, c) = 0$
- $p_1 < c \Rightarrow u_1(p_1, c) < 0$ (given $\alpha > c$, so that $\alpha > p_1$)
- $p_1 > c \Rightarrow u_1(p_1, c) = 0$

Thus

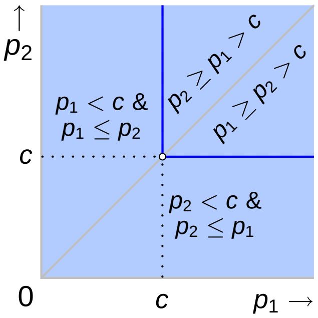
$$u_1(c, c) \geq u_1(p_1, c) \text{ for all } p_1$$

and similarly for firm 2, so (c, c) is a Nash equilibrium

Example of Bertrand's duopoly

Proof that no pair $(p_1, p_2) \neq (c, c)$ is Nash equilibrium

- $p_1 < c$ and $p_1 \leq p_2$? No: $u_1(p_1, p_2) < 0$ and $u_1(c, p_2) = 0$, so firm 1 can profitably deviate to c
- $p_2 < c$ and $p_2 \leq p_1$? No: firm 2 can profitably deviate to c
- $p_1 = c$ and $p_2 > c$? No: firm 1 can profitably *raise* its price: $u_1(c, p_2) = 0$ and $u_1(p_1, p_2) > 0$ for $c < p_1 < p_2$ and $p_1 < \alpha$
- $p_2 = c$ and $p_1 > c$? No: similar reason
- $p_i \geq p_j > c$? No: firm i can increase its profit by lowering p_i to slightly below p_j if $D(p_j) > 0$ (i.e. if $p_j < \alpha$) and to p^m if $D(p_j) = 0$ (i.e. if $p_j \geq \alpha$)



Methods for finding Nash equilibria: Summary

Appropriate method depends on the game

Exhaustive Check every action profile

Best responses Find best response function of every player and solve for an equilibrium

Exploration + proof Isolate possible equilibria based on exploration of the game, then prove that you have found all equilibria

Finding Nash equilibria: Example

	L	C	R
T	2, 2	1, 2	3, 1
M	3, 0	2, 1	1, 0
B	1, 4	0, 0	2, 3

- *Regardless of player 2's action*, T is better than B for player 1
- We say B is *strictly dominated* by T for player 1
- B is not a best response of player 1 to any action of player 2 $\Rightarrow$ is not used in any Nash equilibrium
- So when looking for Nash equilibria, we can eliminate B from consideration

Strictly dominated actions

Definition

In a strategic game $\langle N, (A_i), (\succsim_i)\rangle$, player i's action $b_i \in A_i$ strictly dominates her action $b'_i \in A_i$ if

$$(a_{-i}, b_i) \succ_i (a_{-i}, b'_i) \text{ for } \textit{every} \text{ list } a_{-i} \text{ of other players' actions,}$$

where $\succ_i$ is player i's strict preference relation.

Strictly dominated actions and Nash equilibrium

- If an action strictly dominates the action b_i', we say that b_i' is strictly dominated
- A strictly dominated action is not a best response to any actions of the other players (whatever the other players do, the action that strictly dominates it is better)
- So a strictly dominated action is not used in any Nash equilibrium
- Thus when looking for Nash equilibria, we can ignore all strictly dominated actions

Finding Nash equilibria: Example

	L	*C*	*R*
T	2, 2	1, 2	3, 1
M	3, 0	2, 1	1, 0
~~*B*~~	~~1, 4~~	~~0, 0~~	~~2, 3~~

- *B* is strictly dominated by *T*
- Thus an action pair is a Nash equilibrium of the game if and only if it is a Nash equilibrium of

	~~*L*~~	*C*	~~*R*~~
~~*T*~~	~~2, 2~~	~~1, 2~~	~~3, 1~~
M	~~3, 0~~	2, 1	~~1, 0~~

- In this game, *C* strictly dominates *R*
- Thus having eliminated *B*, we can eliminate *R*
- Now *M* strictly dominates *T*
- Finally, *C* strictly dominates *L*

Strictly dominated actions and Nash equilibrium

Example

	L	C	R
T	$2,2$	$1,2$	$2,1$
M	$3,0$	$2,1$	$1,0$
B	$1,4$	$0,0$	$1,3$

Conclusion: Unique Nash equilibrium of game is (M, C)

Lessons:

- after a strictly dominated action is eliminated, actions that were not previously strictly dominated may become strictly dominated
- every Nash equilibrium survives *iterative elimination of strictly dominated actions*

Strictly dominated actions and Nash equilibrium

Example

	L	C	R
T	$2,2$	$1,2$	$2,1$
M	$3,0$	$2,1$	$1,0$
B	$1,4$	$0,0$	$1,3$

Conclusion: Unique Nash equilibrium of game is (M, C)

But example is atypical:

- in most games, some action profiles that survive iterated elimination of strictly dominated actions are not Nash equilibria
- in many games, *no* action of any player is strictly dominated

Weakly dominated actions

Player i's action b_i *weakly dominates* her action b'_i if

- b_i is *at least as good* as b'_i for player i regardless of the other players' actions *and*
- b_i is better than b'_i for some list of the other players' actions.

Definition

In a strategic game $\langle N, (A_i)_{i \in N}, (\succsim_i)_{i \in N} \rangle$, player i's action $b_i \in A_i$ weakly dominates her action $b'_i \in A_i$ if

$(a_{-i}, b_i) \succsim_i (a_{-i}, b'_i)$ for *every* list a_{-i} of the other players' actions

and

$(a_{-i}, b_i) \succ_i (a_{-i}, b'_i)$ for *some* list a_{-i} of the other players' actions

Weakly dominated actions: Example

	L	R
T	1, 1	0, 0
B	0, 0	0, 1

- $u_1(T, L) = 1 > 0 = u_1(B, L)$
- $u_1(T, R) = 0 = u_1(B, R)$
- So T weakly dominates B but does not strictly dominate B

Can a weakly dominated action be used by a player in a Nash equilibrium?

Yes! (B, R) is a Nash equilibrium of this game.

Dominated actions: summary

- A strictly dominated action is not a best response to any list of actions of the other players
- So no strictly dominated action is used in a Nash equilibrium
- Every Nash equilibrium survives iterated elimination of strictly dominated actions
- An action that is weakly dominated but not strictly dominated *is* a best response to *some* list of actions of the other players
- A weakly dominated action may be used in a Nash equilibrium

Symmetric games

- Two players
- $A_1 = A_2$
- $(a_1, a_2) \succsim_1 (b_1, b_2)$ if and only if $(a_2, a_1) \succsim_2 (b_2, b_1)$ for all $a \in A$ and $b \in A$
- $\Rightarrow$ there exist payoff representations of preferences such that $u_1(a_1, a_2) = u_2(a_2, a_1)$ for all $a \in A$
- Example:

	L	R
L	w, w	x, y
R	y, x	z, z

Symmetric Nash equilibrium of symmetric game

- Symmetric equilibrium: $a_1^* = a_2^*$
- Does symmetric game necessarily have symmetric equilibrium?
- No:

	L	R
L	$0, 0$	$1, 1$
R	$1, 1$	$0, 0$

- If players are identical, how can *asymmetric* equilibrium be realized?
  - How does a player know which action she should choose?