ECO2030: Microeconomic Theory II, module 1

Lecture 1

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Single person decision problem

Model

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- a set A (the set of *actions*)

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Theory

Single person decision problem

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Theory

Decision-maker chooses $a^* \in A$ that is best according to $\succsim$:

$$a^* \succsim a \text{ for all } a \in A$$

Many decision-makers: Strategic games

Model

A strategic game consists of

Many decision-makers: Strategic games

Model

A strategic game consists of

- a finite set N (the set of *players*)

Many decision-makers: Strategic games

Model

A strategic game consists of

- a finite set N (the set of *players*)

any decision-making entity: individual human being, group of individuals, animal, . . .

Many decision-makers: Strategic games

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A strategic game consists of

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- for each player $i \in N$

Many decision-makers: Strategic games

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- a finite set N (the set of *players*)
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  - a nonempty set A_i (the set of *actions* available to player i)

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any set (numbers, vectors, functions, . . .)

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  - a *preference relation* $\succsim_i$ on $\times_{j \in N} A_j$.

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Action profile is list $(a_j)_{j \in N}$ consisting of one member of each set A_j.

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If $N = \{1, \ldots, n\}$, we write action profile as $(a_1, \ldots, a_n)$.

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Action profile is list $(a_j)_{j \in N}$ consisting of one member of each set A_j.
If $N = \{1, \ldots, n\}$, we write action profile as $(a_1, \ldots, a_n)$.
$\times_{j \in N} A_j$, Cartesian product of collection of sets $(A_j)_{j \in N}$, is set of action profiles.

Many decision-makers: Strategic games

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Each player i has preferences over set of action *profiles*, not only her own actions

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Each player i has preferences over set of action *profiles*, not only her own actions $\implies$ she may care about *every* player's action.
Can represent preference relation by *payoff function*.

Many decision-makers: Strategic games

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Example

- $N = \{1, 2\}$

Example

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- $A_1 = A_2 = \{Q, F\}$

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- $(F, Q) \succ_1 (Q, Q) \succ_1 (F, F) \succ_1 (Q, F)$
  $(Q, F) \succ_2 (Q, Q) \succ_2 (F, F) \succ_2 (F, Q)$

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		Player 2: Q	Player 2: F
Player 1	Q		
	F		

Example

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Player 2

Player 1		Q	F
	Q		
	F	4	

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Player 2

Player 1		Q	F
	Q	3	
	F	4	

Example

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Player 2

		Q	F
Player 1	Q	3	
	F	4	1

Example

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		Player 2: Q	Player 2: F
Player 1	Q	3	0
	F	4	1

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Player 2

Player 1		Q	F
	Q	3	0, 4
	F	4	1

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		Player 2	
		Q	F
Player 1	Q	$3, 3$	$0, 4$
	F	4	$1, 1$

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- Payoff representation isn't unique; any increasing function may be applied separately to each player's payoffs

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Player 2

		Q	F
Player 1	Q	$5, 10$	$1, 11$
	F	$8, -2$	$4, 1$

- Payoff representation isn't unique; any increasing function may be applied separately to each player's payoffs

Example

- $N = \{1, 2\}$
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- Payoff representation isn't unique; any increasing function may be applied separately to each player's payoffs
- Story? Prisoner's Dilemma

Example: BoS

Story

- Two people wish to go out together

Example: BoS

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- The options are concerts of music by

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Example: BoS

	Bach	
Bach		

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- Two people wish to go out together
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Example: BoS

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Story

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Example: BoS

	Bach	*Stravinsky*
Bach		
Stravinsky		

Story

- Two people wish to go out together
- The options are concerts of music by Bach and by Stravinsky

Example: BoS

	Bach	*Stravinsky*
Bach		
Stravinsky		

Bach or Stravinsky?

Story

- Two people wish to go out together
- The options are concerts of music by Bach and by Stravinsky

Example: BoS

	Bach	*Stravinsky*
Bach	2, 1	
Stravinsky		1, 2

Bach or Stravinsky?

Story

- Two people wish to go out together
- The options are concerts of music by Bach and by Stravinsky
- They want to go out together, but one prefers Bach and the other prefers Stravinsky

Example: BoS

	Bach	*Stravinsky*
Bach	$2, 1$	$0, 0$
Stravinsky	$0, 0$	$1, 2$

Bach or Stravinsky?

Story

- Two people wish to go out together
- The options are concerts of music by Bach and by Stravinsky
- They want to go out together, but one prefers Bach and the other prefers Stravinsky
- If they go to different concerts, each of them is equally unhappy listening to the music of either composer

Example: Cournot's oligopoly game

Players $N = \{1, \ldots, n\}$ (firms)

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Actions $A_i = [0, \infty)$ for $i = 1, \ldots, n$ (set of possible outputs)

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Preferences Preferences of each firm are represented by payoff function u_i with

$$u_i(q_1, \ldots, q_n) =$$

Example: Cournot's oligopoly game

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$$u_i(q_1, \ldots, q_n) = q_i P\left(\sum_{j=1}^{n} q_j\right)$$

(firm i's profit), where $P : \mathbb{R}_+ \to \mathbb{R}_+$ ("inverse demand function")

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$$u_i(q_1, \ldots, q_n) = q_i P\left(\sum_{j=1}^{n} q_j\right) - C_i(q_i)$$

(firm i's profit), where $P : \mathbb{R}_+ \to \mathbb{R}_+$ ("inverse demand function") and $C_i : \mathbb{R}_+ \to \mathbb{R}_+$ (firm i's cost function).

Equilibrium

Every player

is rational

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Every player

is rational

$\Updownarrow$

action is best given some belief about other players' actions

Equilibrium

Every player

is rational

$\Updownarrow$

action is best given some belief about other players' actions

belief about other players' actions is correct

Equilibrium

Every player

has played game many times previously, against variety of other players, and knows from her experience what other players will do

is rational

⇕

action is best given some belief about other players' actions

belief about other players' actions is correct

Equilibrium

Every player

is rational

⇕

action is best given some belief about other players' actions

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is rational

⇕

action is best given some belief about other players' actions

belief about other players' actions is correct

↘ ↙

Every player's action is best given other player's actions

Equilibrium

Every player

is rational

$\Updownarrow$

action is best given some belief about other players' actions

belief about other players' actions is correct

Every player's action is best given other player's actions

$\Rightarrow$ Nash equilibrium

Nash equilibrium

a^* is a *Nash equilibrium* if for all $i \in N$

$$a_i^* \text{ is optimal for } i \text{ according to } \succsim_i \text{ given } a_{-i}^*$$

Nash equilibrium

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Definition

A Nash equilibrium of a strategic game $\langle N, (A_i), (\succsim_i) \rangle$ is an action profile $a^* \in \times_{i \in N} A_i$ such that for all $i \in N$

$$(a_{-i}^*, a_i^*) \succsim_i (a_{-i}^*, a_i) \text{ for all } a_i \in A_i.$$

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Definition

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Given action profile b and action a_i of player i, (b_{-i}, a_i) = action profile in which i's action is a_i and action of every other player j is b_j

Prisoner's Dilemma

		Player 2	
		Q	F
Player 1	Q	$3, 3$	$0, 4$
	F	$4, 0$	$1, 1$

Prisoner's Dilemma

Player 2

		Q	F
Player 1	Q	$3, 3$	$0, 4$
	F	$4, 0$	$1, 1$

Check each action pair in turn:

Prisoner's Dilemma

Player 2

Player 1		Q	F
	Q	$3,3$	$0,4$
	F	$4,0$	$1,1$

Check each action pair in turn:

- (Q, Q):

Prisoner's Dilemma

		Player 2	
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Player 1	Q	$3, 3$	$0, 4$
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Check each action pair in turn:

- (Q, Q): not Nash equilibrium because if player 2 chooses Q, player 1 is better off choosing F than choosing Q

Prisoner's Dilemma

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- (Q, F):

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Check each action pair in turn:

- (Q, Q): not Nash equilibrium because if player 2 chooses Q, player 1 is better off choosing F than choosing Q
- (Q, F): not Nash equilibrium because . . .

Prisoner's Dilemma

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Check each action pair in turn:

- (Q, Q): not Nash equilibrium because if player 2 chooses Q, player 1 is better off choosing F than choosing Q
- (Q, F): not Nash equilibrium because ...
- (F, Q):

Prisoner's Dilemma

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- (Q, F): not Nash equilibrium because ...
- (F, Q): not Nash equilibrium because ...
- (F, F):

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Check each action pair in turn:

- (Q, Q): not Nash equilibrium because if player 2 chooses Q, player 1 is better off choosing F than choosing Q
- (Q, F): not Nash equilibrium because . . .
- (F, Q): not Nash equilibrium because . . .
- (F, F): Nash equilibrium because F is at least as good as Q for each player if the other player chooses F

Prisoner's Dilemma

		Player 2	
		Q	F
Player 1	Q	$3,3$	$0,4$
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Check each action pair in turn:

- (Q, Q): not Nash equilibrium because if player 2 chooses Q, player 1 is better off choosing F than choosing Q
- (Q, F): not Nash equilibrium because ...
- (F, Q): not Nash equilibrium because ...
- (F, F): Nash equilibrium because F is at least as good as Q for each player if the other player chooses F

So: unique Nash equilibrium, (F, F)

BoS

	Bach	*Stravinsky*
Bach	$2, 1$	$0, 0$
Stravinsky	$0, 0$	$1, 2$

BoS

	Bach	*Stravinsky*
Bach	2, 1	0, 0
Stravinsky	0, 0	1, 2

- Two Nash equilibria, (*Bach*, *Bach*) and (*Stravinsky*, *Stravinsky*)

BoS

	Bach	*Stravinsky*
Bach	2, 1	0, 0
Stravinsky	0, 0	1, 2

- Two Nash equilibria, (*Bach*, *Bach*) and (*Stravinsky*, *Stravinsky*)
- Note: equilibria are not Pareto ranked

Matching Pennies

	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

Matching Pennies

	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

- No Nash equilibrium!

Example

	L	R
T	$1, 1$	$2, 1$
B	$0, 0$	$2, 4$

Example

	L	R
T	$1, 1$	$2, 1$
B	$0, 0$	$2, 4$

- (T, L):

Example

	L	R
T	$1, 1$	$2, 1$
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- (T, L): Nash equilibrium

Example

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- (T, L): Nash equilibrium
- (T, R):

Example

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- (T, L): Nash equilibrium
- (T, R): Nash equilibrium

Example

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T	$1, 1$	$2, 1$
B	$0, 0$	$2, 4$

- (T, L): Nash equilibrium
- (T, R): Nash equilibrium
- (B, L):

Example

	L	R
T	$1, 1$	$2, 1$
B	$0, 0$	$2, 4$

- (T, L): Nash equilibrium
- (T, R): Nash equilibrium
- (B, L): Not Nash equilibrium

Example

	L	R
T	$1,1$	$2,1$
B	$0,0$	$2,4$

- (T,L): Nash equilibrium
- (T,R): Nash equilibrium
- (B,L): Not Nash equilibrium
- (B,R):

Example

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T	$1, 1$	$2, 1$
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- (B, R): Nash equilibrium

Example

	L	R
T	$1, 1$	$2, 1$
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- (T, L): Nash equilibrium
- (T, R): Nash equilibrium
- (B, L): Not Nash equilibrium
- (B, R): Nash equilibrium

Example: Cournot's model of oligopoly

Players $N = \{1, \ldots, n\}$ (firms).

Actions $A_i = [0, \infty)$ for $i = 1, \ldots, n$ (set of possible outputs).

Preferences Firm i's preferences are represented by payoff function u_i with

$$u_i(q_1, \ldots, q_n) = q_i P\left(\sum_{j=1}^{n} q_j\right) - C_i(q_i)$$

(i's profit), where P is an inverse demand function and C_i is firm i's cost function.

Can't examine every action pair in turn . . . Need a different technique

Best response functions

$B_i(a_{-i}) =$ set of player i's best actions given a_{-i}

Best response functions

$$\begin{aligned} B_i(a_{-i}) &= \text{set of player } i\text{'s best actions given } a_{-i} \\ &= \{a_i \in A_i : (a_{-i}, a_i) \succsim_i (a_{-i}, a_i') \text{ for all } a_i' \in A_i\} \end{aligned}$$

Best response functions

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In terms of payoffs,

$$B_i(a_{-i}) = \arg\max_{a_i} u_i(a_{-i}, a_i)$$

Best response functions

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In terms of payoffs,

$$B_i(a_{-i}) = \arg\max_{a_i} u_i(a_{-i}, a_i)$$

Set of maximizers of $u_i(a_{-i}, a_i)$

Best response functions

$$\begin{aligned} B_i(a_{-i}) &= \text{set of player } i\text{'s best actions given } a_{-i} \\ &= \{a_i \in A_i : (a_{-i}, a_i) \succsim_i (a_{-i}, a_i') \text{ for all } a_i' \in A_i\} \end{aligned}$$

In terms of payoffs,

$$B_i(a_{-i}) = \arg\max_{a_i} u_i(a_{-i}, a_i)$$

Nash equilibrium

$a^* \in \times_{i \in N} A_i$ is a Nash equilibrium if and only if

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Best response functions

Procedure for finding Nash equilibria

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- Problem 1 on Problem Set 1 asks you to use procedure to find Nash equilibria of example of Cournot's model

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3. Prove that no other action profile is an equilibrium

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Players $N = \{1, \ldots, n\}$ (firms)

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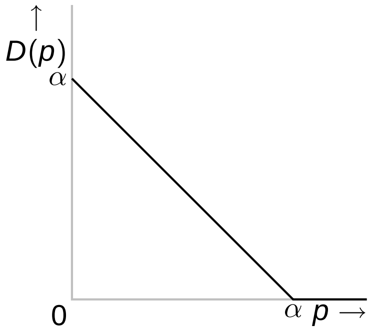

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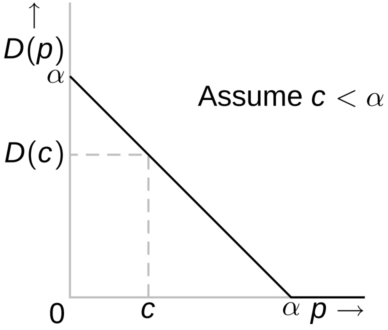

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Assumptions $\Rightarrow$ payoff function of each firm i is

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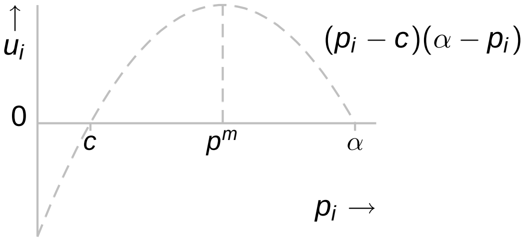

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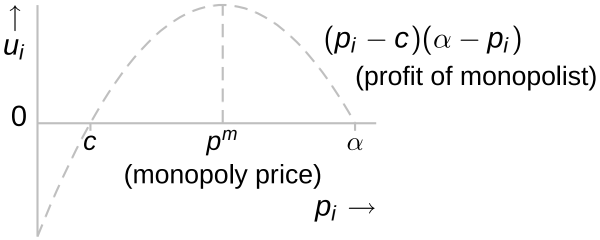

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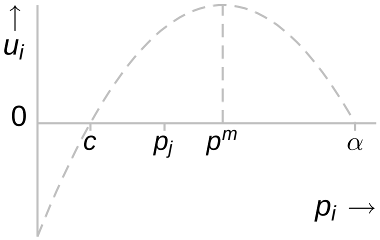

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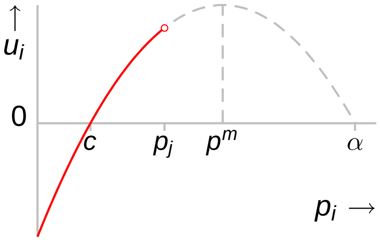

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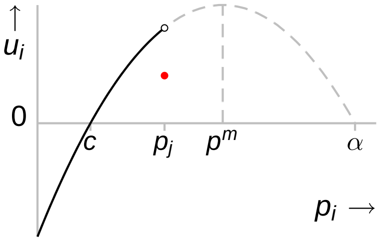

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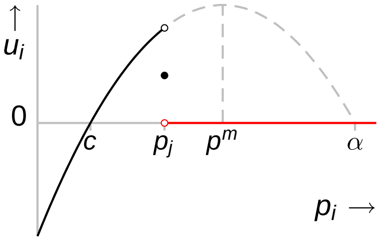

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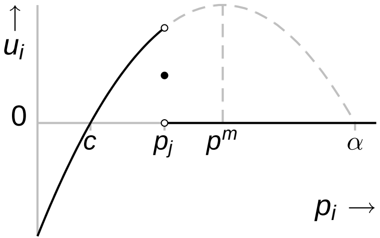

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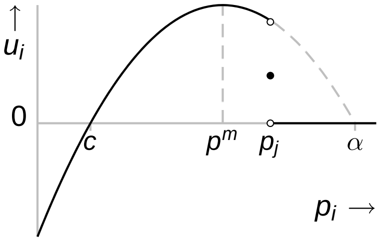

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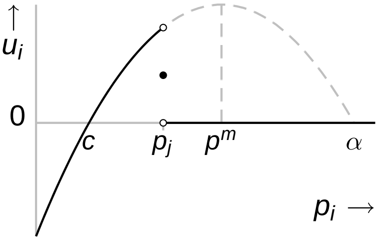
Example of Bertrand's duopoly

Exploration



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Example of Bertrand's duopoly

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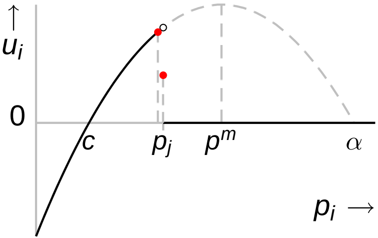

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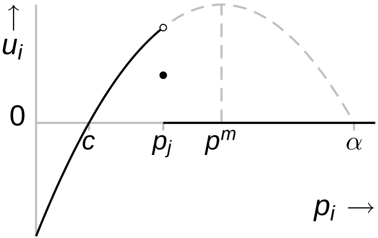

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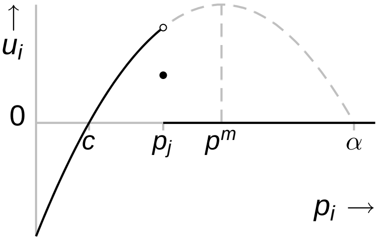

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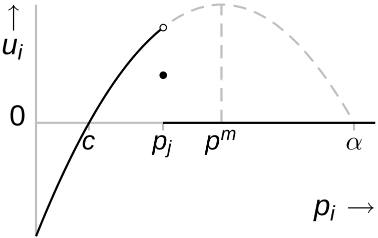
- Conclusion: (c, c) may be only equilibrium?



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Example of Bertrand's duopoly

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Example of Bertrand's duopoly

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$$u_i(p_1, p_2) = \begin{cases} (p_i - c)(\alpha - p_i) & \text{if } p_i < p_j \\ \frac{1}{2}(p_i - c)(\alpha - p_i) & \text{if } p_i = p_j \\ 0 & \text{if } p_i > p_j \end{cases}$$

- $u_1(c, c) = 0$
- $p_1 < c \Rightarrow u_1(p_1, c) < 0$ (given $\alpha > c$, so that $\alpha > p_1$)
- $p_1 > c \Rightarrow u_1(p_1, c) = 0$

Thus

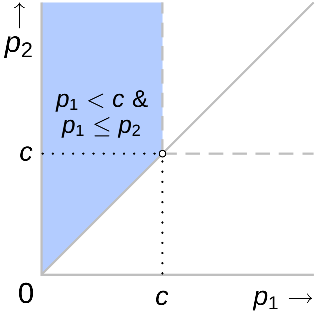
$$u_1(c, c) \geq u_1(p_1, c) \text{ for all } p_1$$

and similarly for firm 2, so (c, c) is a Nash equilibrium

Example of Bertrand's duopoly

Proof that no pair $(p_1, p_2) \neq (c, c)$ is Nash equilibrium

- $p_1 < c$ and $p_1 \leq p_2$?



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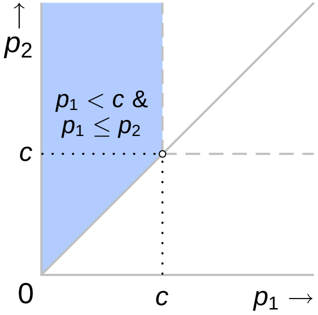
- $p_1 < c$ and $p_1 \leq p_2$? No: $u_1(p_1, p_2) < 0$ and $u_1(c, p_2) = 0$, so firm 1 can profitably deviate to c



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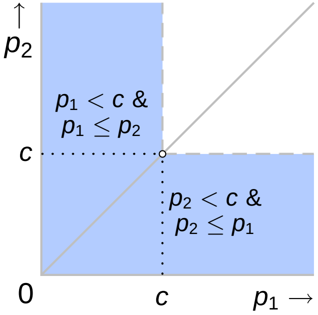

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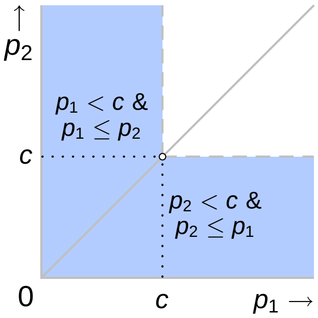

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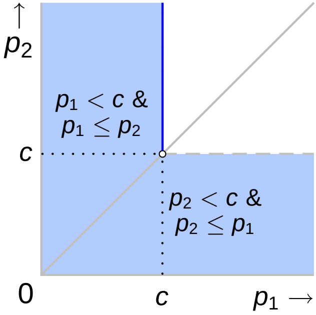

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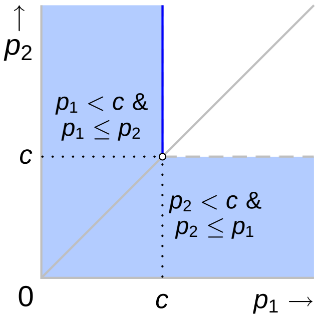

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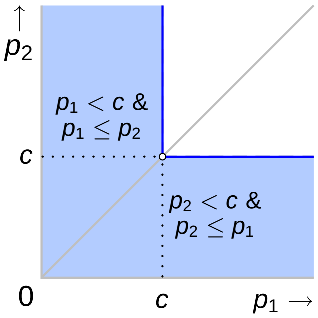

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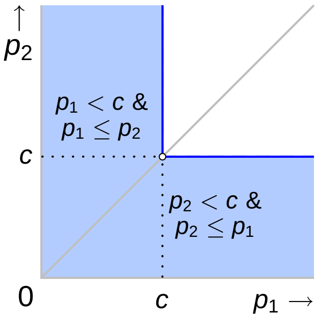

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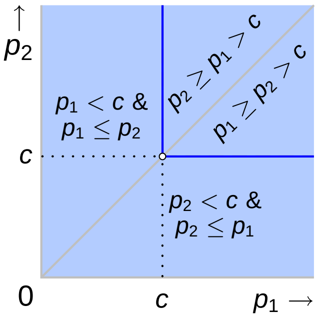
- $p_2 = c$ and $p_1 > c$? No: similar reason
- $p_i \geq p_j > c$? No: firm i can increase its profit by lowering p_i to slightly below p_j if $D(p_j) > 0$ (i.e. if $p_j < \alpha$) and to p^m if $D(p_j) = 0$ (i.e. if $p_j \geq \alpha$)



Methods for finding Nash equilibria: Summary

Appropriate method depends on the game

Exhaustive Check every action profile

Best responses Find best response function of every player and solve for an equilibrium

Exploration + proof Isolate possible equilibria based on exploration of the game, then prove that you have found all equilibria

Finding Nash equilibria: Example

	L	C	R
T	$2, 2$	$1, 2$	$3, 1$
M	$3, 0$	$2, 1$	$1, 0$
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- So when looking for Nash equilibria, we can eliminate B from consideration

Strictly dominated actions

Definition

In a strategic game $\langle N, (A_i), (\succsim_i) \rangle$, player i's action $b_i \in A_i$ strictly dominates her action $b_i' \in A_i$ if

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where $\succ_i$ is player i's strict preference relation.

Strictly dominated actions and Nash equilibrium

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- Finally, C strictly dominates L

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Yes! (B, R) is a Nash equilibrium of this game.

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L	w, w	x, y
R	y, x	z, z

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  - How does a player know which action she should choose?