UNIVERSITY OF TORONTO
Faculty of Arts and Science

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ECO 316H1F (Applied Game Theory)
Instructor: Martin J. Osborne

Duration: 3 hours

No aids allowed

This examination paper consists of **6** pages and **8** questions. Please bring any discrepancy to the attention of an invigilator. The number in brackets at the start of each question is the number of points the question is worth.

Answer all questions.

TO OBTAIN CREDIT, YOU MUST GIVE ARGUMENTS TO SUPPORT YOUR ANSWERS. ALL ANSWERS MUST BE WRITTEN ON THIS PAPER.

1. Two firms produce goods that are partially (not perfectly) substitutable. When the prices set by the firms are p_1 and p_2 (nonnegative numbers), the demand faced by firm 1 is $10 - p_1 + 2p_2$ and the demand faced by firm 2 is $20 - p_2 + \frac{1}{2}p_1$. Each firm's cost of production is zero.

   (a) [3] Model the competition between the firms as a strategic game in which each firm chooses a price.

   Solution: **Players** The two firms

   Actions The set of actions of each firm is the set of nonnegative numbers

   Payoffs When the prices are (p_1, p_2) the payoff of firm 1 is

   $$p_1(10 - p_1 + 2p_2)$$

   and the payoff of firm 2 is

   $$p_1(20 - p_2 + \tfrac{1}{2}p_1).$$

   (b) [7] Find a Nash equilibrium of the game.

   Solution: Firm 1's payoff function is $p_1(10 - p_1 + 2p_2)$, so that its best response function is $b_1(p_2) = 5 + p_2$.
   Firm 2's payoff function is $p_2(20 - p_2 + \frac{1}{2}p_1)$, so that its best response function is $b_2(p_1) = 10 + \frac{1}{4}p_1$.
   A pair of prices (p_1^*, p_2^*) is a Nash equilibrium if and only if $b_1(p_2^*) = p_1^*$ and $b_2(p_1^*) = p_2^*$. Solving these two equations simultaneously yields $(p_1^*, p_2^*) = (20, 15)$. Thus the game has a unique Nash equilibrium, $(20, 15)$.

2. [14] For what ranges of values of the numbers a and b does the following strategic game have a mixed strategy Nash equilibrium in which each player assigns positive probability *only* to her actions A and B? For values of a and b for which the game has such an equilibrium, specify the equilibrium.

	A	B	C
A	$2, 2$	$4, 0$	$4, 1$
B	$4, 1$	$3, 3$	$4, 1$
C	$0, 4$	$a, 4$	$b, 4$

Solution: For player 1's expected payoffs to A and B to be the same, we need $q = \frac{1}{3}$ (the probability that player 2 uses A), and for player 2's expected payoffs to A and B to be the same, we need $p = \frac{1}{2}$ (the probability that player 1 uses A).

This mixed strategy pair is a mixed strategy Nash equilibrium if each player's expected payoff to C is no more than her expected payoff to A and B.

Given $q = \frac{1}{3}$, player 1's expected payoff to A and B is $\frac{10}{3}$ and her expected payoff to C is $\frac{2}{3}a$. Thus for the game to have a mixed strategy Nash equilibrium in which player 1 assigns positive probabilities only to A and B we need $\frac{2}{3}a \le \frac{10}{3}$, or $a \le 5$.

Given $p = \frac{1}{2}$, player 2's expected payoff to A and B is $\frac{3}{2}$ and her expected payoff to C is 1.

Thus the game has a mixed strategy Nash equilibrium in which each player assigns positive probability only to A and B if and only if $a \le 5$; there is no restriction on b.

3. [12] Consider the citizen-candidate model of electoral competition.

[Reminder: In the citizen-candidate model, a position is a number. There is a continuum of citizens, each of whom has a favorite position. The distribution of favorite positions has a unique median, denoted m. Each citizen chooses whether to stand as a candidate; the citizens make their decisions simultaneously. Each citizen votes for the candidate whose position is closest to the citizen's favorite position. The candidate who obtains the most votes wins. (If t candidates are tied for the largest number of votes, each of them wins with probability $1/t$.) If no citizen stands as a candidate, every citizen's payoff is $K < b - c$. Otherwise, each citizen's payoff is the negative of the absolute value of the distance between her favorite position and the position of the winner, minus c if she is a candidate and plus pb if she is a candidate and wins with probability p.]

Assume $0 \le b < c$. Find the set of values of d such that the game has a Nash equilibrium in which exactly one candidate enters and does so at $m - d$.

Solution: The payoff of a citizen who is the only candidate to enter and does so at $m - d$ is $b - c$. If she exits, her payoff is K, which is less than $b - c$ by assumption.

Consider a citizen whose favorite position x satisfies $|m - x| > d$. If this citizen enters, she loses and does not affect the outcome, so it is optimal for her to stay out.

Finally, consider a citizen whose favorite x satisfies $|m - x| < d$. If this citizen enters, she wins and obtains the payoff $b - c$. The citizen with the most to gain is the one whose favorite position is close to $m + d$. Her payoff if she does not enter is close to $-2d$ (the position of the winner is $m - d$) and her payoff if she enters is $b - c$ (because in this case she wins). Thus her payoff from entering is higher than her payoff from staying out if $b - c > -2d$. Thus if $d \le \frac{1}{2}(c - b)$, no citizen can gain by entering.

We conclude that the game has a Nash equilibrium in which a single citizen with favorite position $m - d$ enters if $0 \le d \le \frac{1}{2}(c - b)$.

4. [10] Consider Cournot's model of duopoly in which the inverse demand function is defined by $P(Q) = \alpha - Q$ if $Q \le \alpha$, 0 otherwise, and the cost function of firm i is q_i^2 for $i = 1, 2$. (Note the square in the cost function.)

   Is the output $\alpha/3$ strictly dominated? Either find an output that strictly dominates it or show that no other output strictly dominates it.

Solution: Yes, it is strictly dominated by the output $\alpha/4$, the monopoly output. For $\alpha - q_i - q_j \ge 0$ (so that the price is nonnegative), the payoff of firm i is

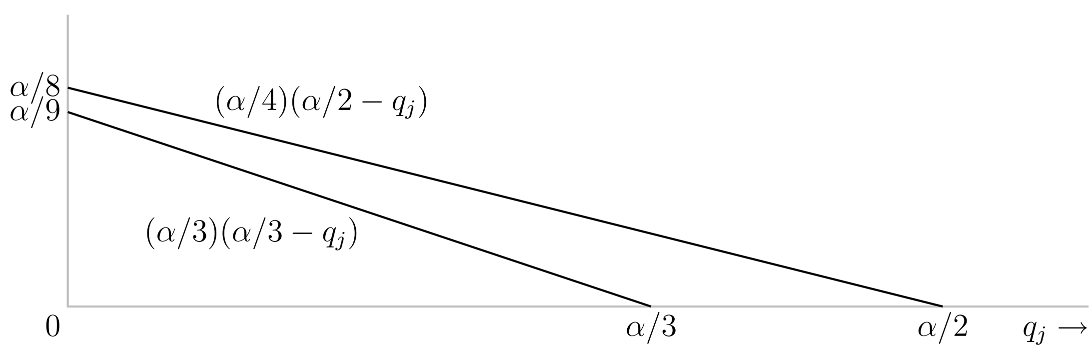
$$\begin{cases} q_i(\alpha - q_i - q_j) - q_i^2 = q_i(\alpha - 2q_i - q_j) & \text{if } q_j \le \alpha - q_i \\ -q_i^2 & \text{if } q_j > \alpha - q_i \end{cases}$$

(where j is the other firm) so its payoff if $q_i = \alpha/3$ is

$$\begin{cases} (\alpha/3)(\alpha/3 - q_j) & \text{if } q_j \le 2\alpha/3 \\ -\alpha^2/9 & \text{if } q_j > 2\alpha/3 \end{cases}$$

and its payoff if $q_i = \alpha/4$ is

$$\begin{cases} (\alpha/4)(\alpha/2 - q_j) & \text{if } q_j \le 3\alpha/4 \\ -\alpha^2/16 & \text{if } q_j > 3\alpha/4. \end{cases}$$



The second payoff exceeds the first for all values of q_2, so the output $\alpha/4$ strictly dominates the output $\alpha/3$.

5. Consider a first-price "all-pay" sealed-bid auction with *three* bidders.

Each bidder knows her own valuation, but not the other bidders' valuations; she believes that each of the other bidder's valuations is uniformly distributed from 0 to 1. Call a bidder with valuation v a "bidder of type v".

The bidder who submits the highest bid is the winner. (You can ignore ties.) If a bidder of type v bids b and wins the auction then her payoff is $v - b$; if she does not win her payoff is $-b$. (That is, she pays b **whether or not she wins**.)

(a) [8] Suppose that for some number $\beta > 0$ the bid of each type v_2 of player 2 is βv_2^3 and the bid of each type v_3 of player 3 is βv_3^3. (In both cases, note that the valuations are *cubed*.) Find the expected payoff of type v_1 of player 1 when she bids b_1, for any $0 \le v_1 \le 1$.

Solution: Given the bidding functions of players 2 and 3, each of their bids is uniformly distributed from 0 to β. Thus

- if player 1 bids more than β she wins and obtains the payoff $v_1 - b_1$
- if player 1 bids at most β then the probability with which she wins is the probability that *both* of the other players' bids are less than b_1, which is $(b_1/\beta)^{1/3}(b_1/\beta)^{1/3} = (b_1/\beta)^{2/3}$, so that her expected payoff is $(b_1/\beta)^{2/3}v_1 - b_1$.

So the payoff of type v_1 of player 1 when she bids b_1 is

$$\begin{cases} (b_1/\beta)^{2/3}v_1 - b_1 & \text{if } b_1 \le \beta \\ v_1 - b_1 & \text{if } b_1 > \beta. \end{cases}$$

(b) [7] Does the auction have an equilibrium in which the bid of each player i is βv_i^3 for some value of $\beta > 0$? (Again, note the *cube*.) If so, find the value of β.

Solution: The maximizer of $(b_1/\beta)^{2/3}v_1 - b_1$ satisfies $\frac{2}{3}b_1^{-1/3}v_1/\beta^{2/3} - 1 = 0$, or $b_1 = (\frac{2}{3}v_1/\beta^{2/3})^3$.
In particular, if $\beta = (\frac{2}{3})^3/\beta^2$, or $\beta = \frac{2}{3}$, then player 1's optimal bidding function is $\frac{2}{3}v_1^3$. The same considerations apply to the other players, so the auction has a Nash equilibrium in which the bid of each player i is $\frac{2}{3}v_i^3$.

6. (a) [7] Does the ultimatum game have a Nash equilibrium with an outcome that differs from the outcome of the unique subgame perfect equilibrium? Either specify such an equilibrium (be sure to give a pair of **strategies**) or show that no such equilibrium exists. [Reminder: the ultimatum game is a two-player extensive game with perfect information in which player 1 chooses an amount of money x, with $0 \le x \le 1$, to offer to player 2, and then player 2 either accepts the offer, in which case player 1's payoff is $1 - x$ and player 2's payoff is x, or rejects the offer, in which case both players' payoffs are zero.]

Solution: Yes: For example, for any number $z \in (0, 1]$ player 1 offers z and player 2's strategy accepts any offer of at least z and rejects all other offers.

(b) [10] Consider an extension of the ultimatum game in which if player 2 rejects an offer of player 1, she makes a counteroffer, which player 1 can either accept or reject. Formulating a counteroffer takes some time, and the size of the pie available after player 2 rejects player 1's initial offer is smaller than it is initially — $k < 1$ rather than 1.

More precisely, suppose that player 1's offer in the first period is x ($0 \leq x \leq 1$). If player 2 accepts this offer, then the game ends and the payoff of player 1 is $1 - x$ and the payoff of player 2 is x. If player 2 rejects player 1's offer, then player 2 choose an amount y to make as a counteroffer ($0 \leq y \leq k$, where $0 < k < 1$). If player 1 accepts this counteroffer, then the game ends and the payoff of player 1 is y and the payoff of player 2 is $k - y$. If player 1 rejects player 2's offer, the game ends and both players' payoffs are zero.

Find all the subgame perfect equilibria of this extensive game with perfect information. (Be sure to specify the players' **strategies**!)

Solution: The subgame starting with a counteroffer by player 2 is an ultimatum game with pie size k. So it has a unique subgame perfect equilibrium in which player 2 offers 0 and player 1 accepts all offers. The outcome of this equilibrium is that player 1's payoff is 0 and player 2's payoff is k.

In the subgame following an offer of x by player 1, if player 2 accepts the offer her payoff is x whereas if she rejects it her payoff is k. Thus she optimally accepts the offer if $x > k$, rejects it if $x < k$, and is indifferent between accepting and rejecting the offer if $x = k$. So in any subgame perfect equilibrium she either accepts an offer x if and only if $x \geq k$ or accepts an offer if and only if $x > k$. In the first case, player 1's best offer is k, which player 2 accepts, so that the players' payoffs are $1 - k$ and k. In the second case, player 1 has no optimal offer.

Thus the game has a unique subgame perfect equilibrium in which

- player 1's strategy is to offer k at the start of the game and accept all offers after any history in which player 2 rejected player 1's initial offer
- player 2's strategy is to accept an offer x of player 1 if and only if $x \geq k$ and to counteroffer 0 if she rejects player 1's initial offer.

7. [10] Find the range of values of the discount factor δ (in terms of a and b) for which the strategy pair in which both players use the strategy *Unrelenting punishment* is a Nash equilibrium of the following *Prisoner's Dilemma*, where $a > b > 1$. [The strategy *Unrelenting punishment* selects the action C initially and after any history in which the other player chose C in every previous period; after any other history, it selects D.]

	C	D
C	b, b	$0, a$
D	$a, 0$	$1, 1$

How does the range change as a increases?

Solution: The condition for a player not to want to deviate is

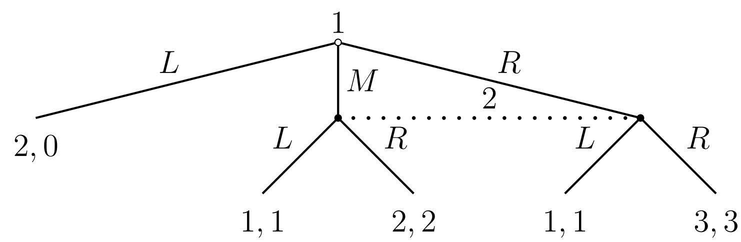
$$\frac{b}{1-\delta} \geq a + \frac{\delta}{1-\delta}$$

or

$$\delta \geq \frac{a-b}{a-1}.$$

As a increases, this lower bound increases. That is, if the payoff to D increases then the players need to be more patient to sustain cooperation as a Nash equilibrium.

8. Consider the following extensive game (with imperfect information).



(a) [3] Does the game have a Nash equilibrium in which player 2 chooses L? If so, specify the equilibrium.

Solution: If player 2 chooses L then player 1's best action is L. And if player 1 chooses L then L is optimal for player 2. (R is also optimal for player 2.) So (L, L) is a Nash equilibrium of the game.

(b) [4] Does the game have a weak sequential equilibrium in which player 2 chooses L?

Solution: For every belief of player 2 at her information set, the only optimal action is R, so the game has no weak sequential equilibrium in which player 2 chooses L.

(c) [5] Does the game have a weak sequential equilibrium in which player 2 chooses R?

Solution: If player 2 chooses R then player 1's optimal action is R. Consequently in a weak sequential equilibrium player 1's belief at her information set is $(0, 1)$ (that is, she assigns probability 1 to R), in which case R is optimal for her. Thus the game has a weak sequential equilibrium in which the strategy pair is (R, R) and player 2's belief assigns probability 1 to R.