ECO316: Applied game theory

Lecture 12

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Table of contents

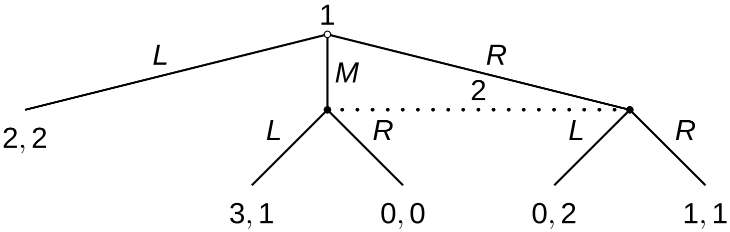
- Extensive games
  - Perfect recall
- Strategies
- Nash equilibrium
- Beliefs and assessments
- Weak sequential equilibrium
  - Example 1
  - Example 2
- Signaling games
  - Entry
  - Entry
  - Education

Extensive games with imperfect information

- Extensive game with perfect information: players are perfectly informed about *past* actions
- Now consider games in which players are *not* perfectly informed about past actions
- Example: firm may know whether another firm has entered its industry, but may not know how much the entrant has decided to spend on R & D

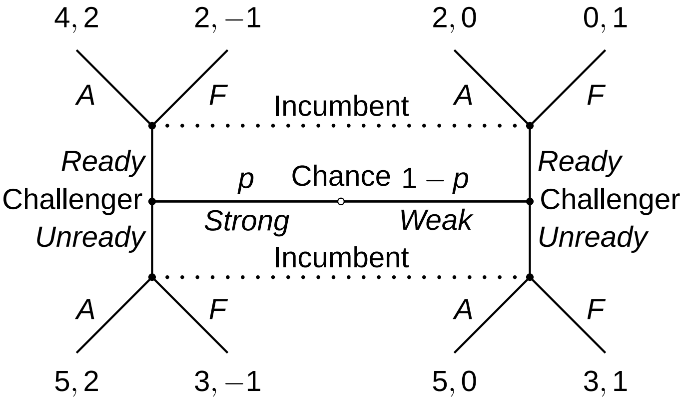
Extensive games with imperfect information

Example



Extensive games with imperfect information

Example with chance move



Extensive games with imperfect information

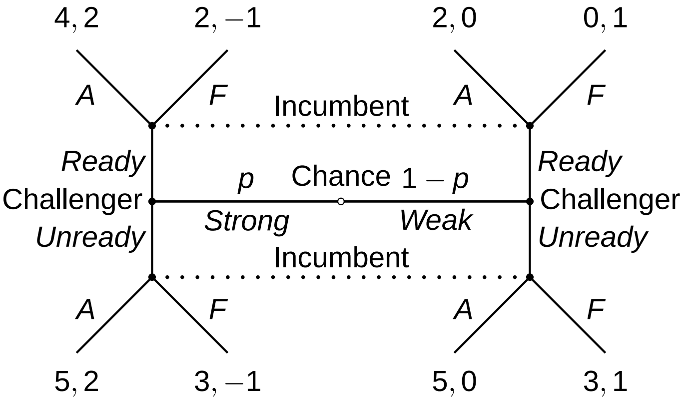
Definition

An extensive game consists of

- A set N of players
- A set H of histories
- A player function P that specifies the player, or chance, who moves after each nonterminal history
- A specification of the probabilities used by chance
- A specification of the set of histories among which each player cannot distinguish at each point at which she moves (the player's *information sets*)
- Preferences over terminal histories for each player

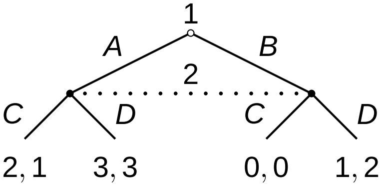
Extensive games with imperfect information

Example

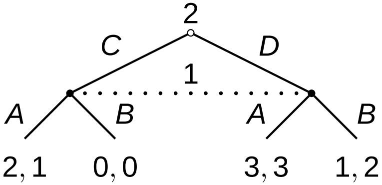


Extensive games with imperfect information

Example

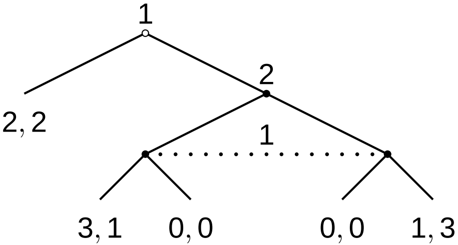


Models same situation as strategic game in which players 1 and 2 choose actions simultaneously, as does following game



Extensive games with imperfect information

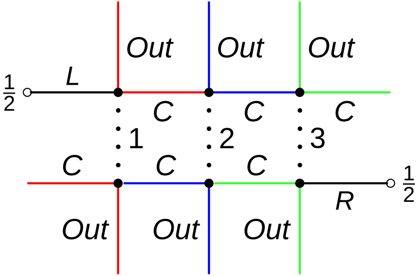
Example



Models situation in which player 1 moves and then players 1 and 2 move simultaneously

Extensive games with imperfect information

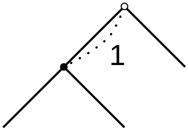
Example



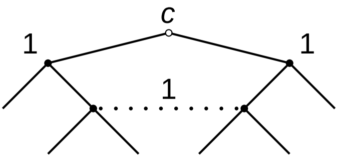
- Player 1 does not know whether she is the first mover or whether she is moving after the other players have moved
- Player 2 does not know whether she is moving after player 1 and before player 3, or the other way around
- Player 3 does not know whether she is the first mover or whether she is moving after the other players have moved

Extensive games with imperfect information

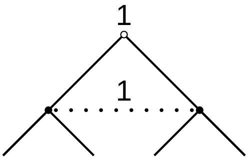
Examples



Player does not know whether she is choosing action at start of game, or whether she has already chosen an action



When choosing at her last information set, player does not know move of chance, which she knew at start of game



When making her second choice, player does not know action she chose at start of game

These games have *imperfect recall*

Extensive games with imperfect information

Perfect recall

- Game has perfect recall if at every point every player remembers whatever she knew in the past
- Will restrict throughout to games with perfect recall

Strategies

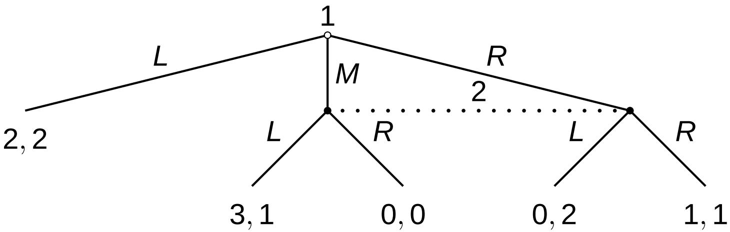
Definition

A pure strategy of a player in an extensive game is a function that assigns an action to every information set at which the player moves

Given set of strategies for each player, can define strategic form of extensive game as before

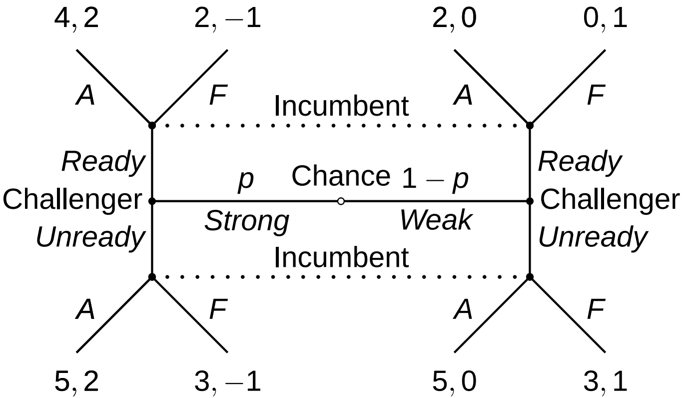
Strategic form

Example



	L	R
L	2, 2	2, 2
M	3, 1	0, 0
R	0, 2	1, 1

Strategies



	AA	AF	$FA \cdots FF$
RR	$4p + 2(1-p), 2p$	$4p, 2p + 1 - p$	...
RU	$4p + 5(1-p), 2p$	$4p + 3(1-p), 2p + 1 - p$	...
UR	$5p + 2(1-p), 2p$	$5p, 2p + 1 - p$	...
UU	$5, 2p$	$5p + 3(1-p), 2p + 1 - p$	...

Nash equilibrium

Definition

A Nash equilibrium of an extensive game is a Nash equilibrium of the strategic form of the game

- Alternatively, we can define Nash equilibrium directly in terms of the extensive game
- For any strategy profile s, let $O(s)$ be the outcome of s: the terminal history generated by s

Definition

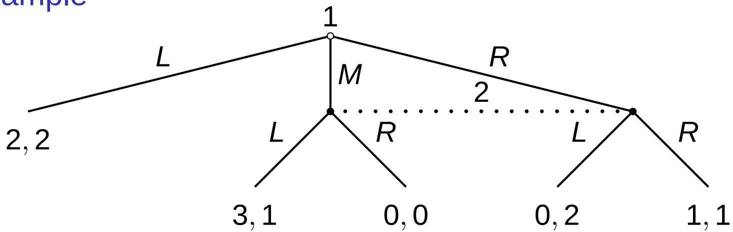
A Nash equilibrium of an extensive game is a strategy profile s such that for every player $i \in N$

$$O(s^*_{-i}, s^*_i) \succsim_i O(s^*_{-i}, s_i) \text{ for every strategy } s_i \text{ of player } i$$

That is, no player is better off deviating from her strategy, given the other players' strategies

Nash equilibrium

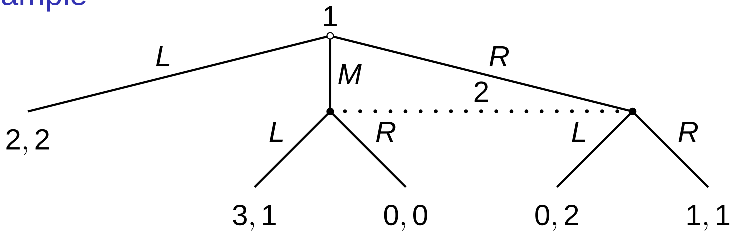
Example



- One Nash equilibrium: (M, L)
- Player 1's payoff: 3
- Player 1's payoff to deviating to L: 2
- Player 1's payoff to deviating to R: 0
- Player 2's payoff: 1
- Player 2's payoff to deviating to R: 0

Nash equilibrium

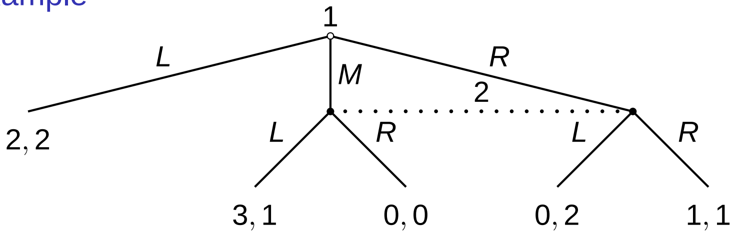
Example



- Another Nash equilibrium: (L, R)
- Player 1's payoff: 2
- Player 1's payoff to deviating to M: 0
- Player 1's payoff to deviating to R: 1
- Player 2's payoff: 2
- Player 2's payoff to deviating to L: 2

Nash equilibrium

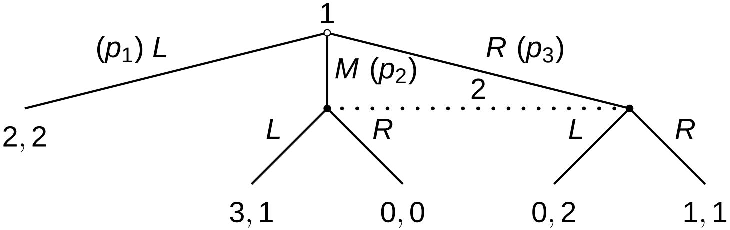
Example



- Another Nash equilibrium: (L, R)
  - But if player 1 deviates to M or R, player 2's action L is better than R *regardless of whether she believes player 1 chose M or R*
  - Like incumbent's action *Fight* in NE (*Out*, *Fight*) of Entry game, player 2's strategy R is not optimal if player 2's information set is reached
  - But this NE *is* subgame perfect, because the game has no proper subgame
  - We need new refinement of NE

Nash equilibrium

Example



- In this game, optimal action of player 2 is L regardless of her belief about whether player 1 chose M or R
- But for other payoffs, optimal action depends on her belief
- If player 1 chooses M and/or R with positive probability, player 2's belief can be derived from player 1's strategy
- But if player 1 chooses L, player 2's belief *cannot* be derived from player 1's strategy
  - Need to specify player 2's belief as part of equilibrium

Beliefs and assessments

A belief system for an extensive game is a function that assigns to every information set a probability distribution over the set of histories in the information set

- Probability distribution assigned to information set I represents beliefs of player $P(I)$ who moves at I about probabilities of histories in I

An assessment in an extensive game is pair (s, μ) where s is a strategy profile and μ is a belief system

Equilibrium

Sequential rationality
Each player's strategy is optimal given her beliefs

Consistency of beliefs
The belief system is consistent with the strategy profile

Sequential rationality

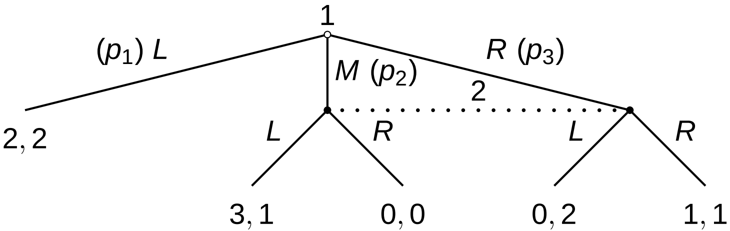
An assessment is sequentially rational if for every information set of every player, the player's strategy is a best response to the other players' strategies given the player's belief at the information set

Weak consistency

An assessment is weakly consistent if for every information set reached with positive probability given the strategy profile, the probability assigned by the beliefs to each history in the information set is derived by using Bayes' rule

Equilibrium

Weak consistency



- An assessment in which player 1 chooses L and player 2 holds *any* belief at her information set is weakly consistent: given player 1's strategy, player 2's information set is not reached
- If $p_2 + p_3 > 0$, so that the information set *is* reached, then weak consistency requires that player 2's belief assign probability $p_2/(p_2 + p_3)$ to M and probability $p_3/(p_2 + p_3)$ to R

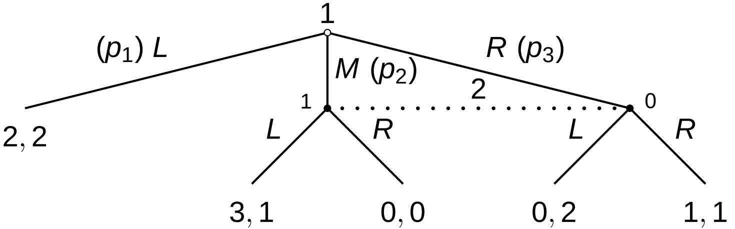
Equilibrium

Definition

An assessment is a weak sequential equilibrium of an extensive game if it is sequentially rational and weakly consistent

Equilibrium

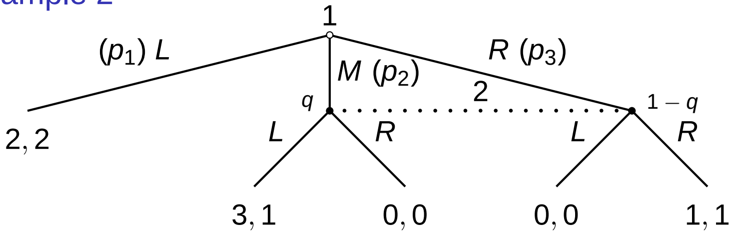
Example 1



- Start by looking at P2's choice
- For *any* belief at P2's information set, only L is optimal
- So in any WSE P2 chooses L
- Given that P2 chooses L, P1's optimal action is M
- What are P2's beliefs at her information set?
- Weak consistency $\Rightarrow q = 1$
- So unique WSE, with strategies (M, L) and beliefs $(1, 0)$

Equilibrium

Example 2



- Start by looking at P2's choice
- If $q > \frac{1}{2}$ then L is only optimal action; if $q < \frac{1}{2}$ then R is only optimal action; if $q = \frac{1}{2}$ then both L and R are optimal
- If P2 chooses L then P1 chooses $M \Rightarrow$ beliefs $(1, 0) \Rightarrow L$ is optimal $\Rightarrow$ assessment $((M, L), (1, 0))$ is WSE
- If P2 chooses R then P1 chooses $L \Rightarrow$ beliefs unrestricted by weak consistency; need $q \leq \frac{1}{2}$ for R to be optimal $\Rightarrow$ any assessment $((L, R), (q, 1-q))$ with $q \leq \frac{1}{2}$ is WSE

Signaling games

- Some parties are informed about variables that affect everyone, some are not informed
- Informed parties can take actions observed by uninformed parties before latter take actions that affect everyone
- Informed parties' actions may "signal" their information
- Example:
  - employer can observe whether job applicant has college degree, but not her ability
  - person of high ability may obtain a degree at low cost, while one of low ability may do so only at high cost
  - then the fact that a person has a degree may signal to an employer that she has high ability—not because college teaches any skills, but because only high-ability individuals find obtaining a degree worthwhile, given the cost

Two-player signaling games

- A *sender* is informed about a variable relevant to both her and a *receiver* who is uninformed
- Sender takes an action observed by receiver
- Receiver takes action that affects them both
- Depending on how message and receiver's action affect the parties, may want to limit or distort information her signal conveys
- Extensive game in which chance move determines sender's value of variable about which sender is informed
- Receiver sees sender's action but not the value of the variable, and then herself takes an action

Signaling games: Entry

- A *challenger* contests an *incumbent*'s turf
- Challenger is well-prepared to fight incumbent (*strong*) with probability p and ill-prepared (*weak*) with probability $1 - p$, where $0 < p < 1$
- Challenger knows its preparedness, but incumbent does not
- Each type of challenger may *ready* itself for battle or remain *unready* (it does not have option of staying out)
- Cost of getting ready for battle is higher for weak challenger
- Incumbent observes challenger's readiness, but not its type, and chooses whether to *fight* or *acquiesce*

Signaling games: Entry

- Incumbent may *acquiesce* to (share market with) challenger or *fight* it
- Acquiesce $\Rightarrow$ payoff P
- Fight weak challenger $\Rightarrow$ win $\Rightarrow$ payoff $M - c$
- Fight strong challenger $\Rightarrow$ lose $\Rightarrow$ payoff $P - c$
- Assume $P = 1$, $M = 3$, $c = 1$

So incumbent's payoff:

$$\begin{cases} 1 & \text{if } \textit{acquiesces} \\ 2 & \text{if } \textit{fights} \text{ and challenger is weak} \\ 0 & \text{if } \textit{fights} \text{ and challenger is strong} \end{cases}$$

Signaling games

Strong challenger's payoff:

$$\begin{cases} \pi & \text{if incumbent } \textit{acquiesces} \text{ and challenger is } \textit{unready} \\ \pi - c_s & \text{if incumbent } \textit{acquiesces} \text{ and challenger is } \textit{ready} \\ \pi - d & \text{if incumbent } \textit{fights} \text{ and challenger is } \textit{unready} \\ \pi - c_s - d & \text{if incumbent } \textit{fights} \text{ and challenger is } \textit{ready} \end{cases}$$

Weak challenger's payoff:

$$\begin{cases} \pi & \text{if incumbent } \textit{acquiesces} \text{ and challenger is } \textit{unready} \\ \pi - c_w & \text{if incumbent } \textit{acquiesces} \text{ and challenger is } \textit{ready} \\ \pi - d & \text{if incumbent } \textit{fights} \text{ and challenger is } \textit{unready} \\ \pi - c_w - d & \text{if incumbent } \textit{fights} \text{ and challenger is } \textit{ready} \end{cases}$$

More costly for a weak challenger that for a strong challenger to prepare for a fight:

$$c_s < c_w$$

Signaling games

Assume $\pi = 5$, $c_s = 1$, $c_w = 3$, and $d = 2$, so that:

Strong challenger's payoff:

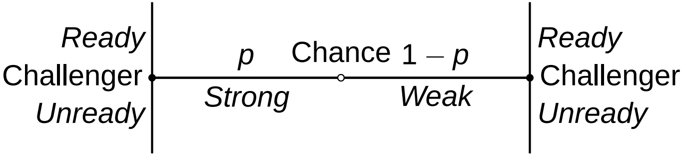
$$\begin{cases} 5 & \text{if incumbent } \textit{acquiesces} \text{ and challenger is } \textit{unready} \\ 4 & \text{if incumbent } \textit{acquiesces} \text{ and challenger is } \textit{ready} \\ 3 & \text{if incumbent } \textit{fights} \text{ and challenger is } \textit{unready} \\ 2 & \text{if incumbent } \textit{fights} \text{ and challenger is } \textit{ready} \end{cases}$$

Weak challenger's payoff:

$$\begin{cases} 5 & \text{if incumbent } \textit{acquiesces} \text{ and challenger is } \textit{unready} \\ 2 & \text{if incumbent } \textit{acquiesces} \text{ and challenger is } \textit{ready} \\ 3 & \text{if incumbent } \textit{fights} \text{ and challenger is } \textit{unready} \\ 0 & \text{if incumbent } \textit{fights} \text{ and challenger is } \textit{ready} \end{cases}$$

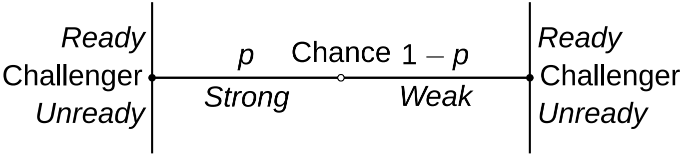
Signaling games: Entry

Challenger is *strong* with probability p and weak with probability $1 - p$



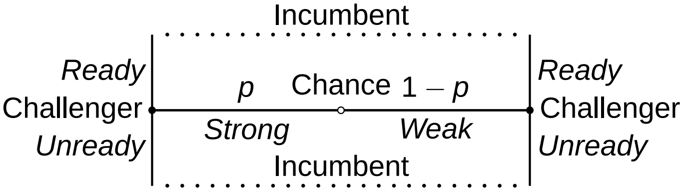
Signaling games: Entry

Challenger knows her type; each type has two actions, *Ready* and *Unready*



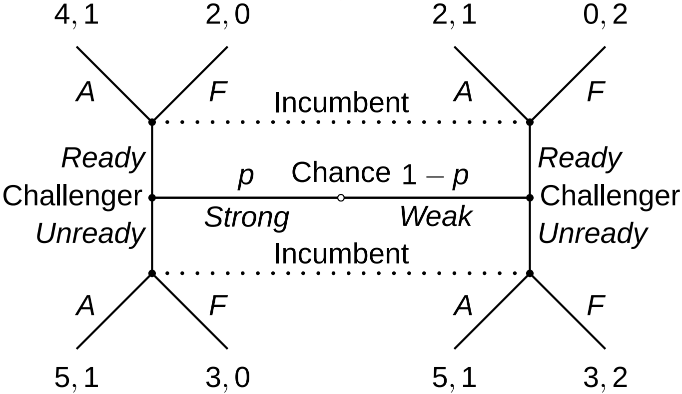
Signaling games: Entry

Incumbent observes only challenger's action, not her type



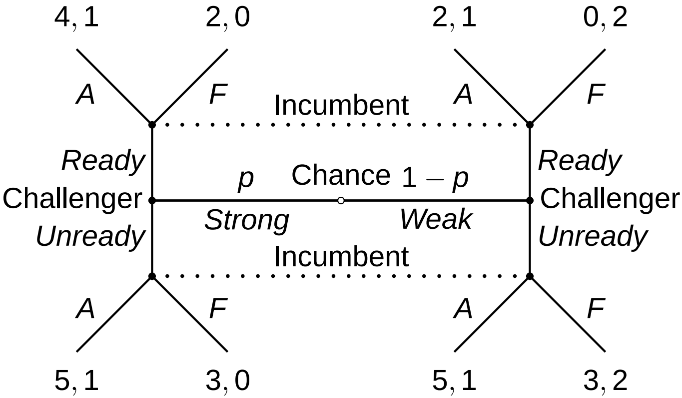
Signaling games: Entry

Incumbent has two actions, *acquiesce* and *fight*



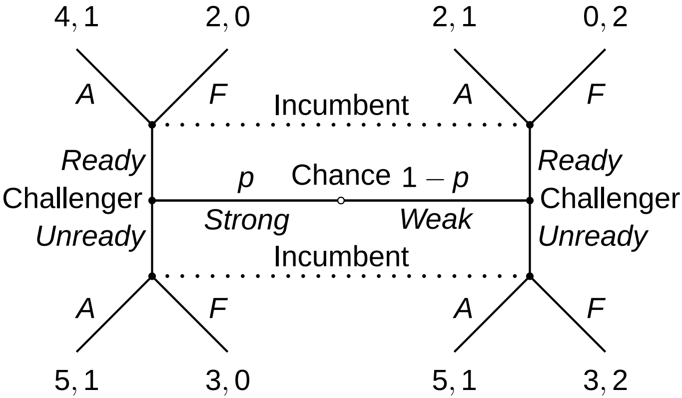
Signaling games: Entry

For weak challenger, cost of becoming *ready* is 3 and $(Ready, Acquiesce)$ is worse than $(Unready, Fight)$

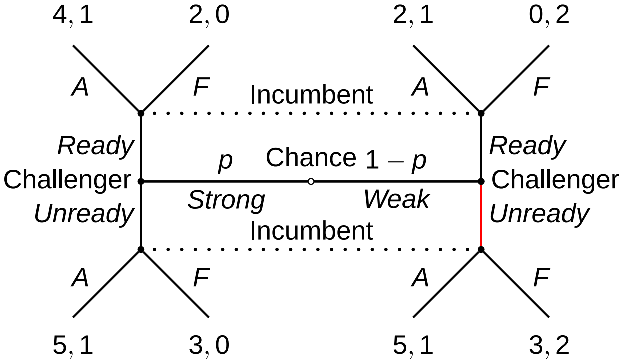


Signaling games: Entry

But incumbent is better off *fighting* than *acquiescing* if challenger is weak

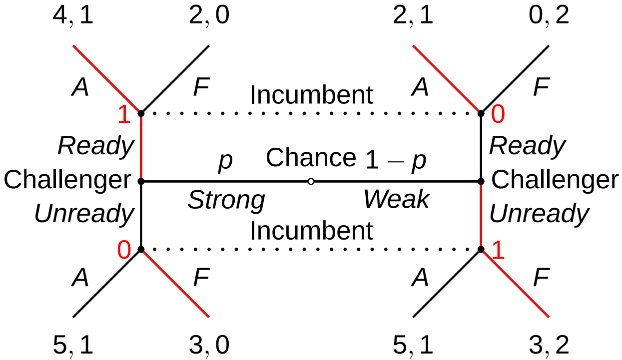


Signaling games: Entry



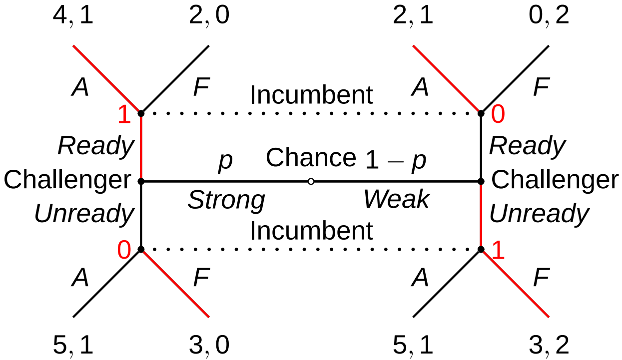
- Weak challenger prefers *Unready* to *Ready* regardless of incumbent's action
- So in any weak sequential equilibrium, weak challenger chooses *Unready*

Signaling games: Entry



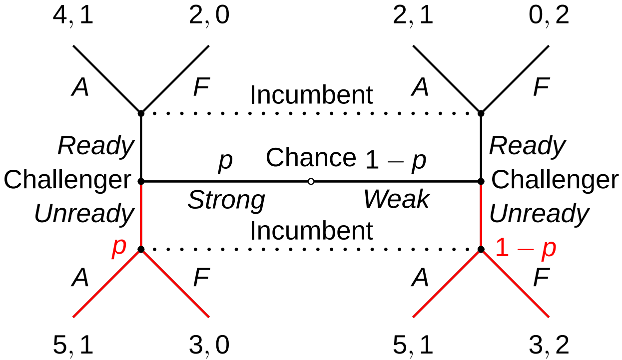
- If strong challenger chooses *Ready*, incumbent must believe *Ready* $\Rightarrow$ *Strong* and *Unready* $\Rightarrow$ *Weak* and hence must choose *A* in response to *Ready* and *F* in response to *Unready*

Signaling games: Entry



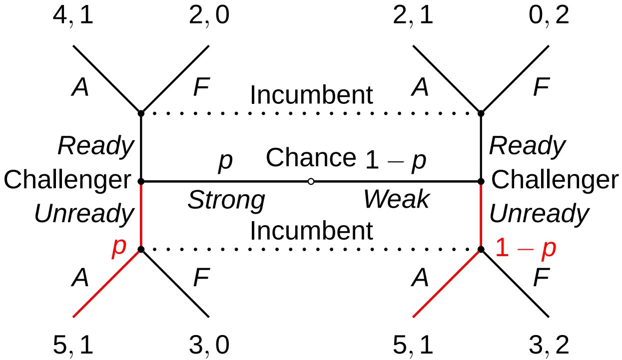
- This assessment is a weak sequential (“separating”) equilibrium: only player remaining to check is strong challenger, who is worse off deviating to *Unready*

Signaling games: Entry



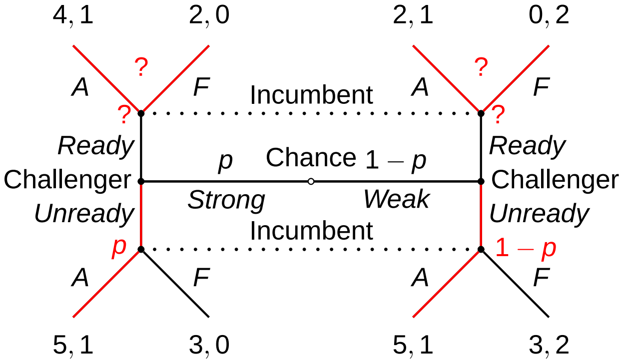
- If strong challenger chooses *Unready*, incumbent must believe *Unready* $\Rightarrow$ *Strong* with probability p and *Weak* with probability $1-p$, so incumbent's payoffs: $A \Rightarrow 1$ and $F \Rightarrow 2(1-p)$
- Hence A optimal if $p \geq \frac{1}{2}$ and F optimal if $p \leq \frac{1}{2}$

Signaling games: Entry



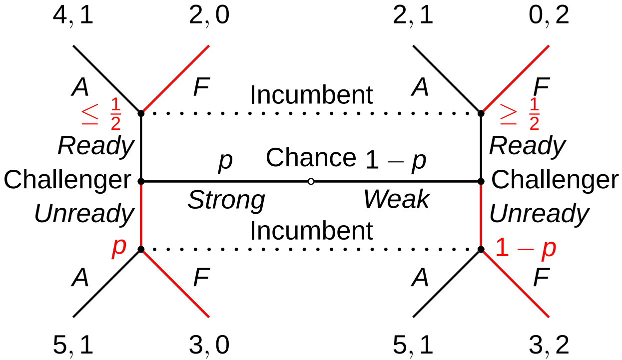
- Given this behavior of incumbent, is *Unready* optimal for strong challenger?
- $p > \frac{1}{2}$: incumbent chooses A, strong challenger who deviates to *Ready* gets at most 4, instead of 5

Signaling games: Entry



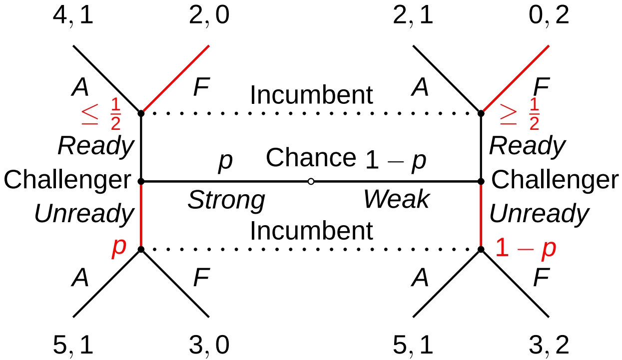
- So for $p \geq \frac{1}{2}$, any such assessment is a ("pooling") equilibrium (with any belief of incumbent after *Ready*)

Signaling games: Entry



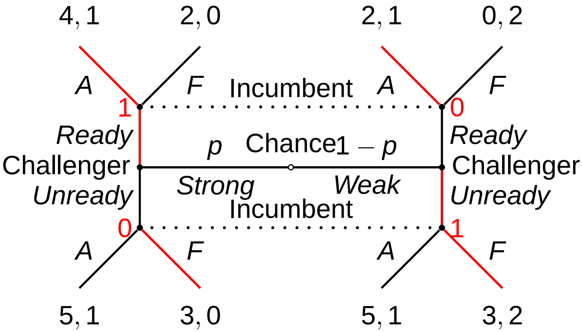
- Given this behavior of incumbent, is *Unready* optimal for strong challenger?
- $p < \frac{1}{2}$: incumbent chooses F, so must choose F also in response to *Ready* (otherwise strong challenger prefers *Ready*), which is optimal if incumbent believes *Ready* challenger is weak with probability $\geq \frac{1}{2}$

Signaling games: Entry



- So if $p < \frac{1}{2}$, any such assessment is a ("pooling") weak sequential equilibrium

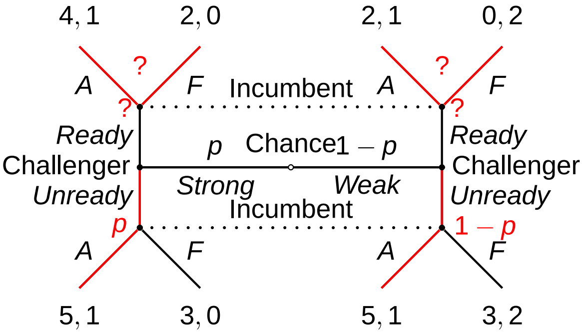
Signaling games: Entry



Separating equilibrium

- Strong challenger chooses *Ready*
- Weak challenger chooses *Unready*
- Incumbent believes *Ready* challenger is strong, *Unready* challenger is weak
- Incumbent fights *Unready* challenger and acquiesces to *Ready* challenger

Signaling games: Entry

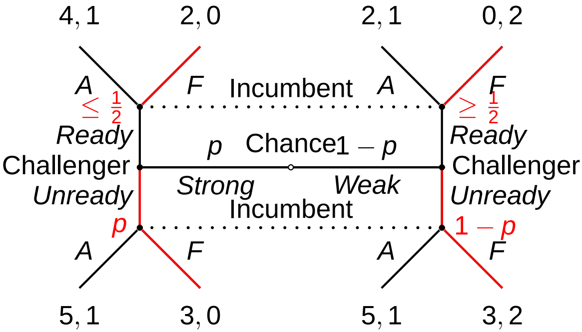


Pooling equilibria

If $p > \frac{1}{2}$,

- both challengers chooses *Unready*
- incumbent believes *Unready* challenger is strong with probability p
- incumbent acquiesces to *Unready* challenger

Signaling games: Entry



Pooling equilibria

If $p < \frac{1}{2}$,

- both challengers choose *Unready*
- incumbent believes *Unready* challenger strong with prob. p and *Ready* challenger is strong with prob. $\leq \frac{1}{2}$
- incumbent fights all challengers

Education as a signal of ability

- Why are you obtaining a college degree?
- Because you think that the principles you learn in your courses will prepare you for the day when you run Google or preside over Italy?
- Possibly—but there may be another reason
- Perhaps nothing you learn in college has any bearing on the job you expect to take, but you need to get a degree to prove to potential employers that your ability is high
- How does your obtaining a degree prove this point?
- Because the cost to persons of low ability of obtaining the degree is much higher than it is for you (they will take longer, and find the process painful), so that such persons cannot profitably imitate you

Education as a signal of ability

- Thus a college degree signals high ability, even if colleges do nothing to foster that ability
- Employers know that a recipient of a college degree must have high ability because only for such a person is it worthwhile to obtain a degree
- If the cost of achieving high proficiency in freestyle snowboarding were much lower for a person with the skills valued by Google or the Italian citizenry than for someone without the skills, a certificate attesting to that achievement could be your ticket to a rewarding job
- But it is not, so you are in college

Education as a signal of ability

Model: structure

- Worker's ability: H or $L < H$
- Two potential employers (firms)
- Worker's ability is known to her but not to potential employers, who believe probability of H is π and probability of L is $1 - \pi$

Worker chooses amount e of education

$\downarrow$

Firms observe e and simultaneously offer wages w_1 and w_2

$\downarrow$

Worker chooses w_1 or w_2

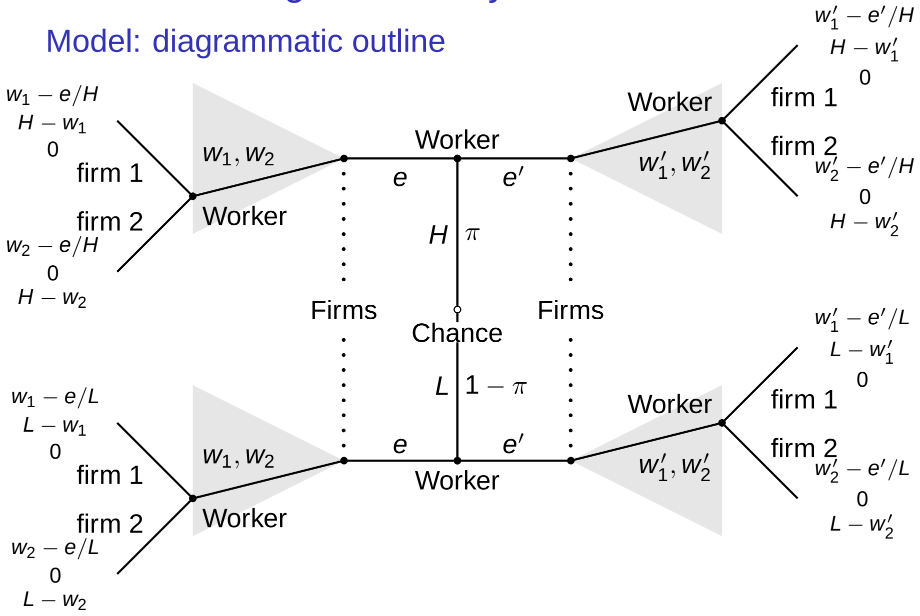
Education as a signal of ability

Model: payoffs

- Education is less expensive for worker of ability H than it is for worker of ability L
- Specifically: cost to worker of ability K of obtaining e units of education is e/K
- So payoff of worker of ability K who obtains e units of education and gets wage w is $w - e/K$
- Payoff of firm that employs worker of ability K at wage w is $K - w$

Education as a signal of ability

Model: diagrammatic outline



Education as a signal of ability

Fix a number $e^* > 0$ and consider the following assessment

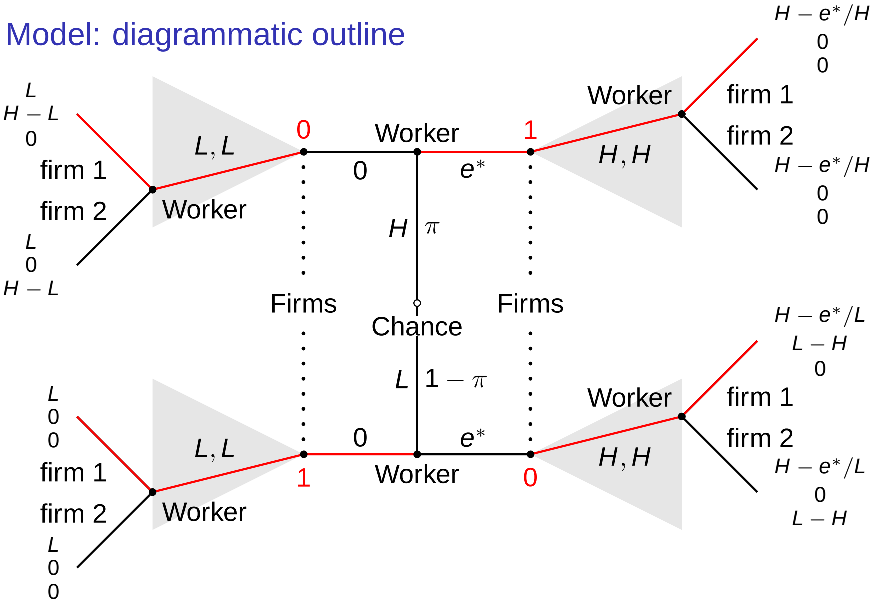
Worker's strategy Type H chooses $e = e^*$ and type L chooses $e = 0$; after observing the firms' wage offers, both types choose the highest offer if they differ, and that of firm 1 if they are the same.

Firms' belief Each firm believes that a worker is type H if she chooses e^* and type L otherwise.

Firms' strategies Each firm offers the wage H to a worker who chooses e^* and the wage L to a worker who chooses *any* other value of e.

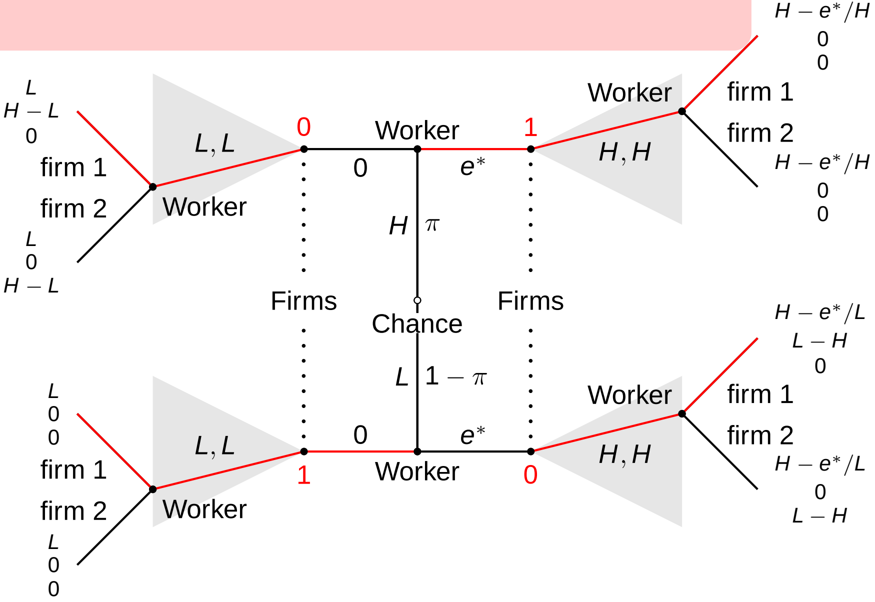
Education as a signal of ability

Model: diagrammatic outline



Is assessment an equilibrium?

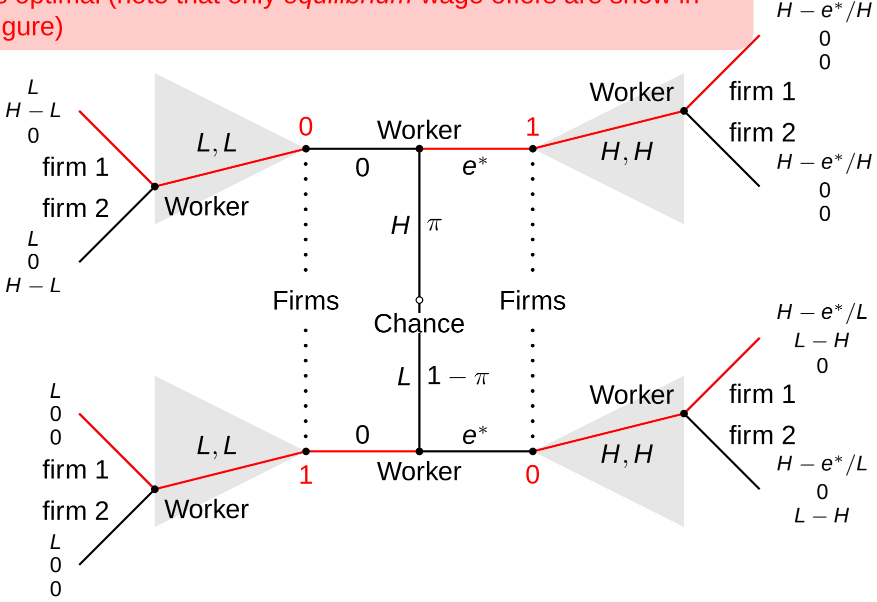
- Firm's beliefs are consistent with worker's strategy



E

Is assessment an equilibrium?

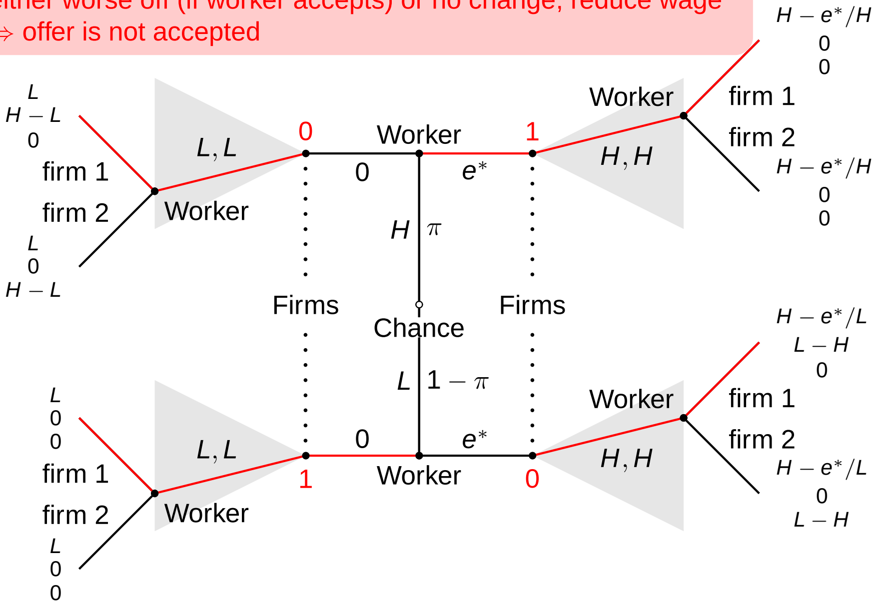
- Worker's wage-acceptance behavior: accepting highest wage is optimal (note that only *equilibrium* wage offers are show in figure)



Is assessment an equilibrium?

- Firm: Given belief and strategy of other firm, raise wage ⇒ either worse off (if worker accepts) or no change; reduce wage ⇒ offer is not accepted

E

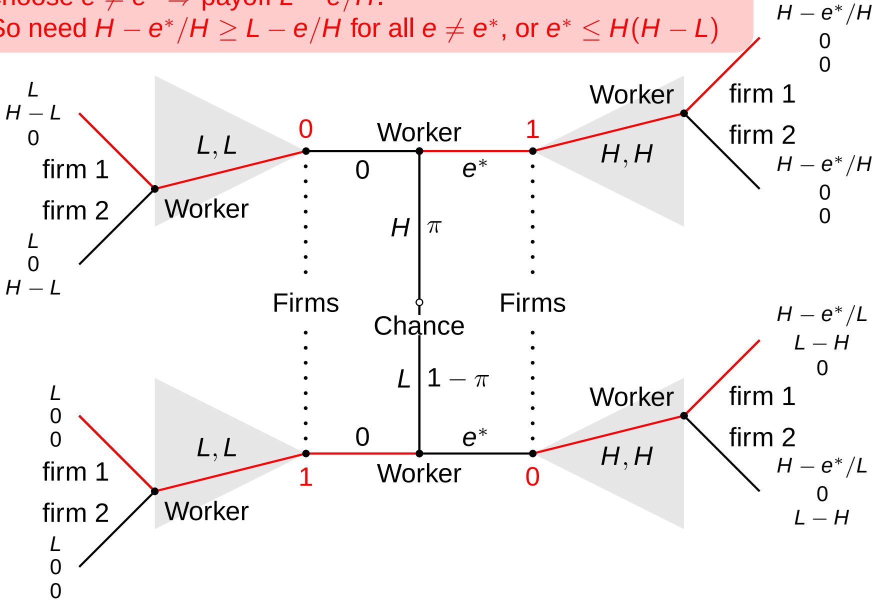


E

Is assessment an equilibrium?

- H worker's choice of e: Choose $e^* \Rightarrow$ payoff $H - e^*/H$; choose $e \neq e^* \Rightarrow$ payoff $L - e/H$.

So need $H - e^*/H \geq L - e/H$ for all $e \neq e^*$, or $e^* \leq H(H - L)$

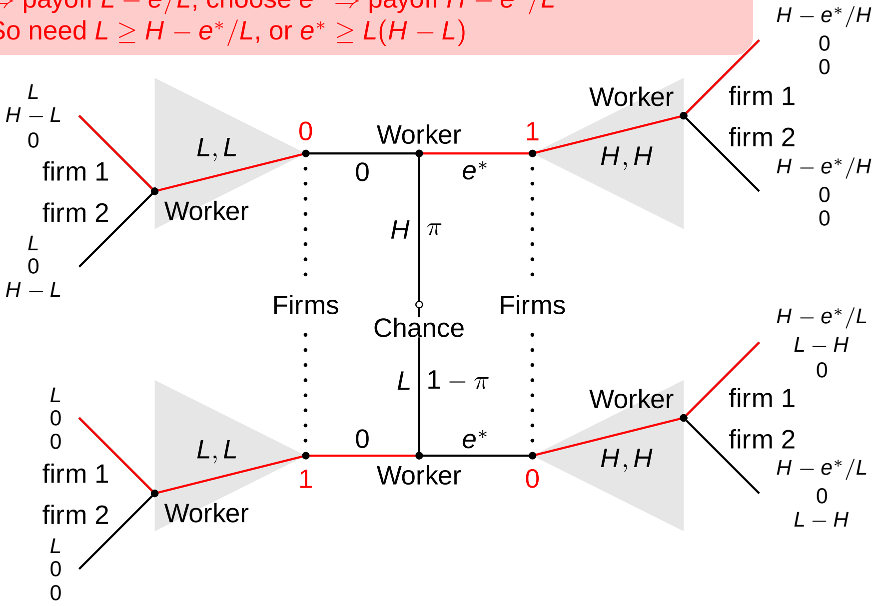


Is assessment an equilibrium?

• L worker's choice of e: Choose 0 $\Rightarrow$ payoff L; choose $e \neq e^*$ $\Rightarrow$ payoff $L - e/L$; choose e^* $\Rightarrow$ payoff $H - e^*/L$

So need $L \geq H - e^*/L$, or $e^* \geq L(H - L)$

E



Education as a signal of ability

Summary

Assessment is a (separating) equilibrium if and only if

$$L(H - L) \leq e^* \leq H(H - L)$$

That is, if e^* satisfies this condition, the game, in which education has no impact on productivity, has equilibrium in which

- high ability worker gets education e^*
- low ability worker gets no education
- firms deduce that education $e^* \Rightarrow$ high ability

Game *also* has (pooling) equilibrium in which all workers get the same amount of education (exercise)