ECO316: Applied game theory

Lecture 11

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2017.11.23

Table of contents

Repeated games: Experiment

- Each participant will be matched with another participant at beginning of experiment
- No one will know the identity of the person with whom they are matched
- Each pair will play a Prisoner's Dilemma repeatedly
- Prisoner's Dilemma interpreted as duopoly, with actions *high price* and *low price*

	high price	low price
high price	$7, 7$	$0, 10$
low price	$10, 0$	$3, 3$

- After each play of game, each participant will be informed of his/her opponent's action
- Number of periods determined randomly
- After first 10 periods, game will end with probability 0.07 after each period

Repeated games

- *Same* set of players repeatedly play strategic game—which we call *stage game*
- Each player observes other players' previous actions
- So each player can condition her action in period t on action profiles in periods $1, \ldots, t-1$
- Play continues indefinitely, but players discount future payoffs
    - players may be "impatient" (prefer to consume now rather than later)
    - game may end with positive probability after any period
- Model allows us to think about long-term relationships and reputation

Repeated games: Payoffs

- Assume players discount future payoffs
- Player i's payoff function in stage game: u_i
- Player i's payoff to sequence $a^1, a^2, a^3, \ldots$ of action profiles in stage game:

$$u_i(a^1) + \delta u_i(a^2) + \delta^2 u_i(a^3) + \cdots$$

  where $0 < \delta < 1$
- Example: suppose outcome in stage game is the *same* in every period: $a, a, a, \ldots$
  Then payoff in repeated game is

$$\begin{aligned} u_i(a) + \delta u_i(a) + \delta^2 u_i(a) + \cdots &= (1 + \delta + \delta^2 + \cdots) u_i(a) \\ &= \frac{1}{1-\delta} u_i(a) \end{aligned}$$

Repeated games: Strategies

Strategy in repeated game is function that specifies

- action in each period *for every possible history of action profiles*

Thus strategy is function that specifies

- action in first period
- action in second period for every possible action profile in first period
- action in third period for all possible action profiles in first and second periods
- and so on ...

Repeated games: Strategies

Example: Stage game is *Prisoner's Dilemma*

	C	D
C	$2, 2$	$0, 3$
D	$3, 0$	$1, 1$

Some possible strategies in repeated game:

All-C Choose C in the first period, and in every future period, for every possible history

All-D Choose D in the first period, and in every future period, for every possible history

C-D alternation Choose C in the first period, and then alternate between D and C, regardless of the action profiles previously chosen

Repeated games: Strategies

Example: Stage game is *Prisoner's Dilemma*

	C	D
C	$2, 2$	$0, 3$
D	$3, 0$	$1, 1$

Some possible strategies in repeated game:

Unrelenting punishment Choose C in first period, and in every future period t choose

$$\begin{cases} C & \text{if other player chose } C \text{ in every period } 1, \ldots, t-1 \\ D & \text{if other player chose } D \text{ in any period } 1, \ldots, t-1 \end{cases}$$

This strategy indefinitely "punishes" a player who deviates from C by choosing D subsequently

Repeated games: Strategies

Example: Stage game is *Prisoner's Dilemma*

	C	D
C	$2, 2$	$0, 3$
D	$3, 0$	$1, 1$

Some possible strategies in repeated game:

k-period punishment Deviation to D induces k periods of punishment

- Two "states": *collude*, *punish*
- Choose C in state *collude*, D in state *punish*
- In period 1, state is *collude*
- Switch from *collude* to *punish* if opponent chooses D
- Switch from *punish* to *collude* after k consecutive periods in which state is *punish*

Strategy "punishes" a player who deviates from C by playing D for k periods

Repeated games: Strategies

Example: Stage game is *Prisoner's Dilemma*

	C	D
C	$2, 2$	$0, 3$
D	$3, 0$	$1, 1$

Some possible strategies in repeated game:

Tit-for-tat Choose C in first period, and in every future period choose action chosen by other player in previous period

Axelrod's experiments

- In late 1970s, Robert Axelrod (political scientist, Michigan) invited submission of strategies (in computer code) for repeated Prisoner's Dilemma
- Invited 14 submissions by social scientists and mathematicians
- Added a strategy that chooses C and D randomly
- Played every strategy against every other 5 times in 200-period game (strategies could have random elements, so could generate different results in different plays)
- Winner: *tit-for-tat* (submitted by Anatol Rapoport (1911–2007), then at U of T)

Axelrod's experiments

- Subsequently, in early 1980s, Axelrod conducted second tournament
- Advertised for submissions in journals read by microcomputer users
- 62 submissions
- Strategies pitted against each other in repeated game of random length
- Winner: *tit-for-tat* (submitted, again, by Anatol Rapoport)
- Axelrod wrote book *The evolution of cooperation* in which he argued the evolutionary merits of tit-for-tat
- Despite *tit-for-tat*'s success and Axelrod's arguments, theoretical results showing that *tit-for-tat* is robustly good are few

Repeated games: Nash equilibrium

Prisoner's Dilemma

	C	D
C	$2,2$	$0,3$
D	$3,0$	$1,1$

- Is strategy pair $(\textit{All-C}, \textit{All-C})$ a Nash equilibrium of infinitely repeated Prisoner's Dilemma?
- Given player 2 uses *All-C*, player 1 uses *All-C* $\Rightarrow$ outcome (C, C) in every period $\Rightarrow$

$$\text{payoff} = 2 + 2\delta + 2\delta^2 + \cdots = \frac{2}{1-\delta}$$

- Player 1 deviates to *All-D* $\Rightarrow$ outcome (D, C) in every period $\Rightarrow$

$$\text{payoff} = 3 + 3\delta + 3\delta^2 + \cdots = \frac{3}{1-\delta} > \frac{2}{1-\delta}$$

- So $(\textit{All-C}, \textit{All-C})$ is not a Nash equilibrium

Repeated games: Nash equilibrium

Prisoner's Dilemma

	C	D
C	$2,2$	$0,3$
D	$3,0$	$1,1$

- Is strategy pair $(All\text{-}D, All\text{-}D)$ a Nash equilibrium of infinitely repeated Prisoner's Dilemma?
- Given player 2 uses *All-D*, player 1 uses *All-D* $\Rightarrow$ outcome (D, D) in every period $\Rightarrow$

$$\text{payoff } = 1 + \delta + \delta^2 + \cdots = \frac{1}{1-\delta}$$

- Player 1 deviates to any other strategy $\Rightarrow$ outcome (C, D) or (D, D) every period $\Rightarrow$ payoff either 0 or 1 in every period $\Rightarrow$

$$\text{payoff} \leq 1 + \delta + \delta^2 + \cdots = \frac{1}{1-\delta}$$

- So $(All\text{-}D, All\text{-}D)$ is a Nash equilibrium

Repeated games: Nash equilibrium

Prisoner's Dilemma

	C	D
C	$2, 2$	$0, 3$
D	$3, 0$	$1, 1$

- Does repeated game have Nash equilibrium that generates better outcome than (D, D) in every period?
- Suppose player 2 uses *Unrelenting punishment* strategy: Choose C in first period, and in every future period t choose

$$\begin{cases} C & \text{if other player chose } C \text{ in } \textit{every} \text{ period } 1, \ldots, t-1 \\ D & \text{if other player chose } D \text{ in } \textit{any} \text{ period } 1, \ldots, t-1 \end{cases}$$

- What is a best response of player 1?

Repeated games: Nash equilibrium

Prisoner's Dilemma

	C	D
C	$2, 2$	$0, 3$
D	$3, 0$	$1, 1$

Is *All-C* a best response to *Unrelenting punishment*?

- Player 1 uses *All-C* $\Rightarrow$ outcome is (C, C) in every period $\Rightarrow$ payoff 2 in every period

Period	1	2	3	...
Outcome	(C, C)	(C, C)	(C, C)	...
Payoff of player 1	2	2	2	...

- Any strategy that generates a different outcome chooses D after some history $((C, C), \ldots, (C, C))$
- Say player 1 chooses C up to $t - 1$, then D in period t $\Rightarrow$

Period	1	...	$t-1$	t	$t+1$	$t+2$	...
Outcome	(C, C)	...	(C, C)	(D, C)	$(C \text{ or } D, D)$	$(C \text{ or } D, D)$	...
Payoff of player 1	2	...	2	3	≤ 1	≤ 1	...

Repeated games: Nash equilibrium

Prisoner's Dilemma

Is *All-C* a best response to *Unrelenting punishment*?

- Player 1 uses *All-C* $\Rightarrow$ payoff in repeated game

$$= 2 + 2\delta + 2\delta^2 + \dots$$

- Player 1 chooses C up to period $t-1$, then D in period t $\Rightarrow$ payoff in repeated game

$$\leq 2 + 2\delta + 2\delta^2 + \cdots + 2\delta^{t-2} + 3\delta^{t-1} + \delta^t + \delta^{t+1} + \cdots$$

- So deviation is not profitable if

$$\begin{aligned} 2 + 2\delta + \cdots + 2\delta^{t-2} + 3\delta^{t-1} + \delta^t + \delta^{t+1} + \dots \\ \leq 2 + 2\delta + \cdots + 2\delta^{t-2} + 2\delta^{t-1} + 2\delta^t + 2\delta^{t+1} + \dots \end{aligned}$$

Repeated games: Nash equilibrium

Prisoner's Dilemma

Is *All-C* a best response to *Unrelenting punishment*?

$$2 + 2\delta + \cdots + 2\delta^{t-2} + 3\delta^{t-1} + \delta^{t} + \delta^{t+1} + \dots$$
$$\leq 2 + 2\delta + \cdots + 2\delta^{t-2} + 2\delta^{t-1} + 2\delta^{t} + 2\delta^{t+1} + \dots$$

$\Leftrightarrow$

$$3\delta^{t-1} + \delta^{t} + \delta^{t+1} + \cdots \leq 2\delta^{t-1} + 2\delta^{t} + 2\delta^{t+1} + \dots$$

$\Leftrightarrow$ (divide by δ^{t-1})

$$3 + \delta + \delta^{2} + \cdots \leq 2 + 2\delta + 2\delta^{2} + \dots$$

$\Leftrightarrow$

$$1 \leq \delta + \delta^{2} + \dots \quad \Leftrightarrow \quad 1 \leq \frac{\delta}{1-\delta} \quad \Leftrightarrow \quad \delta \geq \tfrac{1}{2}$$

Repeated games: Nash equilibrium

Prisoner's Dilemma

	C	D
C	$2,2$	$0,3$
D	$3,0$	$1,1$

Is *All-C* a best response to *Unrelenting punishment*?

- Conclusion: if $\delta \geq \frac{1}{2}$ then the strategy *All-C* is a best response to the strategy *Unrelenting punishment*
- $\delta \geq \frac{1}{2}$ means the players are sufficiently patient
- Is (*All-C*, *Unrelenting punishment*) a Nash equilibrium in this case?
- Is *Unrelenting punishment* a best response to *All-C*?
- No! If player 1 chooses C regardless of player 2's actions, player 2 is better off choosing D in every period
- So (*All-C*, *Unrelenting punishment*) is not a Nash equilibrium of the repeated game

Repeated games: Nash equilibrium

Prisoner's Dilemma

	C	D
C	$2,2$	$0,3$
D	$3,0$	$1,1$

Stage game

	$\ldots$	*UP*	$\ldots$
All-C	$\ldots$	$2/(1-\delta)$	$\ldots$
UP	$\ldots$	$2/(1-\delta)$	$\ldots$

Player 1's payoffs in repeated game

Best responses to *Unrelenting punishment*

- *All-C* is not the only best response to *Unrelenting punishment*
- Any strategy of player 1 that generates the outcome path $(C,C),(C,C),\ldots$ against *Unrelenting punishment* is also a best response
- *Unrelenting punishment* is such a strategy
- Thus (*Unrelenting punishment*, *Unrelenting punishment*) is a Nash equilibrium of the repeated game if $\delta \geq \frac{1}{2}$

Repeated games: Nash equilibrium

Prisoner's Dilemma

	C	D
C	$2, 2$	$0, 3$
D	$3, 0$	$1, 1$

Summary

- *Unrelenting punishment*: Choose C in first period, and in every future period t choose

$$\begin{cases} C & \text{if other player chose } C \text{ in every period } 1, \ldots, t-1 \\ D & \text{if other player chose } D \text{ in any period } 1, \ldots, t-1 \end{cases}$$

- If $\delta \geq \frac{1}{2}$ then strategy pair (*Unrelenting punishment*, *Unrelenting punishment*) is Nash equilibrium of infinitely repeated Prisoner's Dilemma with payoffs shown above

$\Rightarrow$ "collusion" possible if players sufficiently patient

- For Prisoner's Dilemma with different payoffs, cutoff δ differs

Repeated games: Nash equilibrium

Prisoner's Dilemma

	C	D
C	$2, 2$	$0, 3$
D	$3, 0$	$1, 1$

Less severe punishment?

- Does repeated game have Nash equilibrium with outcome (C, C) in every period in which punishment is less severe?
- Suppose player 2 uses k*-period punishment*:
  - Choose C in first period
  - Continue to choose C until player 1 chooses D
  - Starting in period after player 1 chooses D, choose D for k periods (regardless of player 1's actions)
  - After k periods of choosing D, choose C, but again start a k-period punishment (choosing D) if player 1 chooses D in any period
- Does this strategy deter deviations by player 1 from C?

Repeated games: Nash equilibrium

Prisoner's Dilemma

Less severe punishment?

- If player 1 deviates to D in period t, sequence of outcomes:

Period	1	...	$t-1$	t	$t+1$	$t+2$	...	$t+k$	$t+k+1$
Outcome	(C,C)	...	(C,C)	(D,C)	$(?,D)$	$(?,D)$	...	$(?,D)$	$(?,C)$
Payoff of player 1	2	...	2	3	≤ 1	≤ 1	...	≤ 1	?

- If player 1 chooses C in period $t+k+1$ and subsequently then outcome from then on is (C,C)
- If player 1 chooses D in period $t+k+1$ then new cycle of punishment starts
- If profitable deviation exists, then single deviation to D, with return to C, must be profitable

Repeated games: Nash equilibrium

Prisoner's Dilemma

Less severe punishment?

- For deviation not to be profitable, need

$$
\begin{aligned}
3+\delta+\delta^2+\cdots+\delta^k &\leq 2+2\delta+2\delta^2+\cdots+2\delta^k \\
2+(1+\delta+\delta^2+\cdots+\delta^k) &\leq 2(1+\delta+\delta^2+\cdots+\delta^k) \\
2 &\leq 1+\delta+\delta^2+\cdots+\delta^k \\
2 &\leq \frac{1-\delta^{k+1}}{1-\delta} \\
1-2\delta+\delta^{k+1} &\leq 0
\end{aligned}
$$

- $k=1 \Rightarrow 1-2\delta+\delta^2 \leq 0 \Rightarrow (1-\delta)^2 \leq 0 \Rightarrow$ impossible!
- $k=2 \Rightarrow 1-2\delta+\delta^3 \leq 0 \Rightarrow \delta \geq 0.62$
- $k \uparrow \Rightarrow$ cutoff value of $\delta \downarrow \frac{1}{2}$

Repeated games: Nash equilibrium

Prisoner's Dilemma

	C	D
C	$2,2$	$0,3$
D	$3,0$	$1,1$

Less severe punishment: Conclusion

- For any value of $k \geq 2$, strategy pair in which each player punishes other for k periods in event of deviation is Nash equilibrium of infinitely repeated game if δ is large enough
- Larger $k \Rightarrow$ smaller lower bound on δ
- Mutually desirable outcome (C, C) is sustained by short punishment only if players are relatively patient

Repeated games: Nash equilibrium

Prisoner's Dilemma

	C	D
C	$2, 2$	$0, 3$
D	$3, 0$	$1, 1$

Summary

- Infinitely repeated game has Nash equilibrium in which outcome is (D, D) every period
- If players are sufficiently patient, also has Nash equilibrium in which outcome is (C, C) every period
  - In equilibrium strategy pair, each player "punishes" other for choosing D by playing D for some time
  - If players are more patient, punishment can be shorter

Repeated games: Nash equilibrium

Prisoner's Dilemma

	C	D
C	$2, 2$	$0, 3$
D	$3, 0$	$1, 1$

Other equilibrium outcome paths

- By similar arguments, other outcome paths can generated by Nash equilibria
- Example: (C, C), (D, D), (C, C), (D, D), $\ldots$
- Does any Nash equilibrium generate outcome path (C, D), (C, D), (C, D), $\ldots$?
- No:
  - player 1's payoff is 0 for this path
  - if she chooses *All-D* her payoff is 1 in every period

Repeated games: Nash equilibrium

Prisoner's Dilemma

	C	D
C	$2,2$	$0,3$
D	$3,0$	$1,1$

Other equilibrium outcome paths

- Because *All-D* yields player 1 payoff of at least 1 in every period, her payoff in Nash equilibrium cannot be less than $1/(1-\delta)$

Repeated games: Nash equilibrium

Application: Duopoly

Model: Price-setting firms

- Two firms
- Same constant unit cost: $C_i(q_i) = cq_i$ for $i = 1, 2$
- Demand function: $D(p)$
- Profit of firm i:

$$\pi_i(p_1, p_2) = \begin{cases} (p_i - c)D(p_i) & \text{if } p_i < p_j \\ \frac{1}{2}(p_i - c)D(p_i) & \text{if } p_i = p_j \\ 0 & \text{if } p_i > p_j \end{cases}$$

- Firms interact repeatedly
- Each firm discounts future profits at rate δ, with $0 < \delta < 1$

Repeated games: Nash equilibrium

Application: Duopoly

Model: Price-setting firms

- Denote monopoly price by p^m:

$$p^m \text{ maximizes } (p-c)D(p)$$

- Denote monopoly profit by Π^m
- Strategy:
  - In period 1 choose price p^m
  - In period $t \geq 2$,
    - choose p^m if other firm chose p^m in every previous period
    - choose c if other firm chose a price different from p^m in some previous period

Repeated games: Nash equilibrium

Application: Duopoly

Model: Price-setting firms

- Suppose firm 2 uses the strategy
- If firm 1 uses the strategy, then outcome is (p^m, p^m) in every period $\Rightarrow$ firm 1's profit is $\frac{1}{2}\Pi^m$ in every period
- If firm 1 deviates from strategy in period t, firm 2 chooses price c subsequently, regardless of firm 1's price in period t

$\Rightarrow$ good deviation is price slightly less than p^m

$\Rightarrow$ firm 1's profit is close to Π^m in period t

Period	1	...	$t-1$	t	$t+1$	$t+2$	...
Outcome	(p^m, p^m)	...	(p^m, p^m)	$(p^m - \varepsilon, p^m)$	$(?, c)$	$(?, c)$	...
Firm 1's profit	$\frac{1}{2}\Pi^m$	...	$\frac{1}{2}\Pi^m$	$\approx \Pi^m$	0	0	...

Repeated games: Nash equilibrium

Application: Duopoly

Model: Price-setting firms

- Firm 1's payoff if it uses the strategy:

$$\tfrac{1}{2}\Pi^m + \tfrac{1}{2}\delta\Pi^m + \cdots + \tfrac{1}{2}\delta^{t-2}\Pi^m + \tfrac{1}{2}\delta^{t-1}\Pi^m + \tfrac{1}{2}\delta^t\Pi^m + \ldots$$

- Firm 1's payoff if it deviates to price slightly less than p^m in period t:

$$\approx \tfrac{1}{2}\Pi^m + \tfrac{1}{2}\delta\Pi^m + \cdots + \tfrac{1}{2}\delta^{t-2}\Pi^m + \delta^{t-1}\Pi^m$$

- So deviation is not profitable if

$$\delta^{t-1}\Pi^m \leq \tfrac{1}{2}\delta^{t-1}\Pi^m + \tfrac{1}{2}\delta^t\Pi^m + \ldots$$

$$= \tfrac{1}{2}\delta^{t-1}(\Pi^m + \delta\Pi^m + \ldots) = \tfrac{1}{2}\delta^{t-1}\frac{\Pi^m}{1-\delta}$$

Repeated games: Nash equilibrium

Application: Duopoly

Model: Price-setting firms

So deviation by firm 1 is not profitable if

$$\delta^{t-1}\Pi^m \leq \tfrac{1}{2}\delta^{t-1}\frac{\Pi^m}{1-\delta}$$

or

$$1-\delta \leq \tfrac{1}{2}$$

or

$$\delta \geq \tfrac{1}{2}$$

Repeated games: Nash equilibrium

Application: Duopoly

Conclusion for model with price-setting firms

- If $\delta \geq \frac{1}{2}$ then infinitely repeated game has Nash equilibrium in which each firm charges the monopoly price as long as the other firm does so, and otherwise charges the price c
- That is: implicit collusion is an equilibrium if firms are sufficiently patient
- As for Prisoner's Dilemma, collusive outcome can be supported by strategies with less severe punishments

Repeated games

Summary

- When players repeatedly interact, there are equilibrium outcomes that are very different from the equilibrium outcomes of a one-shot interaction
- In strategy pairs that generate such outcomes, each player's action in any period depend on past actions of other players
- Many equilibria exist, with wide range of outcomes
- Some equilibria correspond to "collusive" outcomes (without any explicit collusion)