Economics 316

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Solutions to Problem Set 10

1. (a) From the answer to question 3 of the problems for Tutorial 2, the best response function of firm 2 is the function b_2 defined by

$$b_2(q_1) = \begin{cases} \frac{1}{4}(\alpha - q_1) & \text{if } q_1 \leq \alpha \\ 0 & \text{if } q_1 > \alpha. \end{cases}$$

Firm 1's subgame perfect equilibrium strategy is the value of q_1 that maximizes $q_1(\alpha - q_1 - b_2(q_1)) - q_1^2$, or $q_1(\alpha - q_1 - \frac{1}{4}(\alpha - q_1)) - q_1^2$, or $\frac{1}{4}q_1(3\alpha - 7q_1)$. The maximizer is $q_1 = \frac{3}{14}\alpha$.

We conclude that the game has a unique subgame perfect equilibrium, in which firm 1's strategy is the output $\frac{3}{14}\alpha$ and firm 2's strategy is its best response function b_2.

The outcome of the subgame perfect equilibrium is that firm 1 produces $q_1^* = \frac{3}{14}\alpha$ units of output and firm 2 produces $q_2^* = b_2(\frac{3}{14}\alpha) = \frac{11}{56}\alpha$ units. In a Nash equilibrium of Cournot's (simultaneous-move) game each firm produces $\frac{1}{5}\alpha$. Thus firm 1 produces more in the subgame perfect equilibrium of the sequential game than it does in the Nash equilibrium of Cournot's game, and firm 2 produces less.

(b) The strategy pair is not a subgame perfect equilibrium because if firm 1's output is different from its Nash equilibrium output in Cournot's model it is not optimal for firm 2 to choose its Nash equilibrium output in Cournot's model. The strategy pair is a Nash equilibrium of the extensive game: given firm 2's strategy, firm 1's strategy is optimal.

(c) The strategy pair is not a subgame perfect equilibrium because if firm 1's output is positive then it is not optimal for firm 2 to choose the output α. The strategy pair is a Nash equilibrium of the extensive game: given firm 2's strategy, if firm 1 produces a positive output then the price is zero and its profit is negative.

2. (a) Suppose that $A_1 = \{T, B\}$, $A_2 = \{L, R\}$, and the payoffs are those given in Figure 1. The strategic game has a unique Nash equilibrium, (T, L), in which player 2's payoff is 1. The extensive game has a unique subgame perfect equilibrium, (B, LR) (where the first component of player 2's strategy is her action after the history T and the second component is her action after the history B). In this subgame perfect equilibrium player 2's payoff is 2.

	L	R
T	1, 1	3, 0
B	0, 0	2, 2

Figure 1. The payoffs for the example in Exercise 2a.

(b) Suppose that $A_1 = \{T, B\}$, $A_2 = \{L, R\}$, and the payoffs are those given in Figure 2. The strategic game has a unique Nash equilibrium, (T, L), in which player 2's payoff is 2. A subgame perfect equilibrium of the extensive game is (B, RL) (where the first component of player 2's strategy is her action after the history T and the second component is her action after the history B). In this subgame perfect equilibrium player 1's payoff is 1. (In all the mixed strategy equilibria, as in the pure strategy Nash equilibrium, player 1's expected payoff exceeds 1.)

	L	R
T	2, 2	0, 2
B	1, 1	3, 0

Figure 2. The payoffs for the example in Exercise 2b.

3. The unique Nash equilibrium of the strategic game is (T, L, L), with payoffs $(1, 1, 1)$.

   In the extensive game, the strategies of player 1 are T and B, a strategy of player 2 is a function that assigns to each of the histories T and B one of player 2's actions L or R, and a strategy for player 3 is a function that assigns to each of the four histories (T, L), (T, R), (B, L), and (B, R) one of player 3's actions L or R.

   The game has two subgame perfect equilibria. In one equilibrium the strategies of the three players are s_1, s_2, and s_3 where

- $s_1 = T$
- $s_2(T) = R$ and $s_2(B) = R$
- $s_3(T, L) = L$, $s_3(T, R) = R$, $s_3(B, L) = L$, $s_3(B, R) = R$.

In the other equilibrium the strategies are s_1', s_2', and s_3' where

- $s_1' = B$
- $s_2'(T) = R$ and $s_2'(B) = R$
- $s_3'(T, L) = L$, $s_3'(T, R) = R$, $s_3'(B, L) = L$, $s_3'(B, R) = R$.

Both equilibria yield the payoffs $(0, 2, 1)$.

Thus player 1 is worse off in each subgame perfect equilibrium than she is in the unique Nash equilibrium of the strategic game.

4. The game has two subgame perfect equilibrium: one in which the proposer's offer is 0 and the responder accepts all offers and one in which the proposer's offer is equal to the smallest monetary unit and the responder rejects the offer of 0 and accepts all other offers.

5. (a) Straightforward.

   (b) i. For any value of b_1, the subgame following the history (A, b_1, Y, A) is an ultimatum game. Its unique subgame perfect equilibrium is the strategy pair in which the official demands $(1-\alpha)y$ and the agent agrees to pay any bribe of at most $(1-\alpha)y$. The outcome in the subgame yields the agent the payoff $\alpha y - b_1 - c$ and the official the payoff $b_1 + (1-\alpha)y$.

   ii. Following any history (A, b_1), the agent obtains $\alpha y - b_1 - c$ if she chooses Y and $-c$ if she chooses N. Thus she chooses Y if $b_1 < \alpha y$ and N if $b_1 > \alpha y$. If $b_1 = \alpha y$, she is indifferent between Y and N.

   iii. Consider the following strategy pair.

   Agent
   - A at start of game.
   - Y after any history (A, b_1) with $b_1 \leq \alpha y$ and N after any history (A, b_1) with $b_1 > \alpha y$.
   - A after any history (A, b_1, Y) with $b_1 \leq b_1^*$ and B after any history (A, b_1, Y) with $b_1 > b_1^*$.
   - Y after any history (A, b_1, Y, A, b_2) with $b_2 \leq (1-\alpha)y$ and N after any history (A, b_1, Y, A, b_2) with $b_2 > (1-\alpha)y$.

Official • b_1^* after the history A.

• $(1-\alpha)y$ after any history (A, b_1, Y, A).

I claim that this strategy pair is a subgame perfect equilibrium. It generates the outcome $(A, b_1^*, Y, A, (1-\alpha)y, Y)$, yielding the agent the payoff $\alpha y - b_1^* - c$ and the official the payoff $b_1^* + (1-\alpha)y$.

We have $b_1^* \leq \alpha y - c$, so the agent cannot increase her payoff by switching from A to B at the start of the game.

If the official reduces the value of b_1, her payoff decreases to $b_1 + (1-\alpha)y$. If she increases the value of b_1, her payoff changes to b_1 if $b_1 \leq \alpha y$ (because the agent responds with Y, then A) and changes to 0 if $b_1 > \alpha y$. Because $b_1^* \geq (2\alpha - 1)y$, the official thus cannot increase her payoff by deviating.

The agent's actions after a history (A, b_1) are optimal because Y generates the payoff $\alpha y - b_1^* - c$ for her and N generates the payoff $-c$.

The agent's actions after a history (A, b_1, Y) are also optimal, because both A and B generate the same payoff, namely $\alpha y - b_1^* - c$.

Finally, the subgame following any history (A, b_1, Y, A) is an ultimatum game, and hence the specified strategies form a subgame perfect equilibrium.

We conclude that the strategy pair is a subgame perfect equilibrium of the whole game.

iv. The game has also a subgame perfect equilibrium in which the agent chooses B at the start of the game. Consider, for example, the strategy pair that differs from the one just described only in that the agent chooses A after every history (A, b_1, Y), the official chooses $b_1 = \alpha y$ after the history A, and the agent chooses B at the start of the game. This strategy pair is a subgame perfect equilibrium.

6. (a) The game is given as follows, where the payoffs are in thousands of pounds.

	Split	*Steal*
Split	50, 50	0, 100
Steal	100, 0	0, 0

The game has three Nash equilibria, (*Split*, *Steal*), (*Steal*, *Split*), and (*Steal*, *Steal*).

(b) The game is illustrated Figure 3.

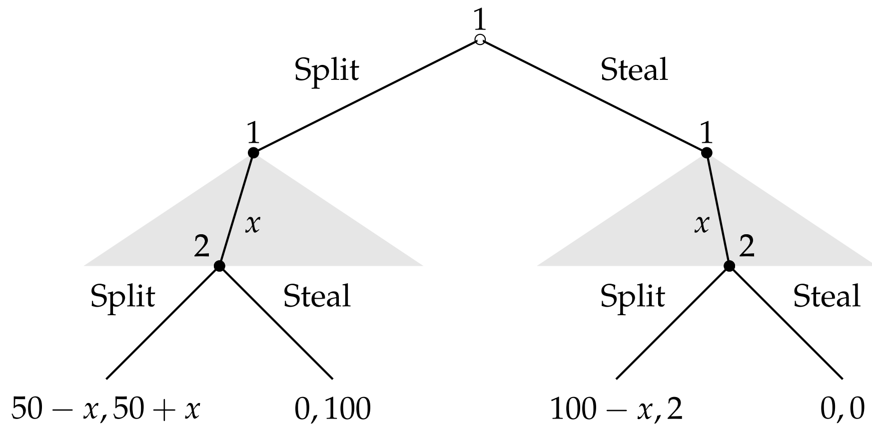

Figure 3. The game in Exercise 6b.

After a history in which player 1 chooses *Split* and a transfer of x, player 2's optimal action is *Steal* if $x < 50$ and either *Split* or *Steal* if $x = 50$.

After a history in which player 1 chooses *Steal* and a transfer of x, player 2's optimal action is *Split* if $x > 0$ and either *Split* or *Steal* if $x = 0$.

Because the subgames following the histories $(Split, 50)$ and $(Steal, 0)$ both have two equilibria, there are four cases to consider for player 1.

- If player 2 chooses *Split* in the subgames following both $(Split, 50)$ and $(Steal, 0)$, then player 1's optimal action is $(Steal, 0)$, following which player 2 chooses *Split* and the payoffs are $(100, 0)$.
- If player 2 chooses *Split* in the subgame following $(Split, 50)$ and *Steal* in the subgame following $(Steal, 0)$, then player 1 has no optimal action (she wants to choose $(Steal, x)$, where x is small, but does not want to choose $(Steal, 0)$.
- If player 2 chooses *Steal* in the subgame following $(Split, 50)$ and *Split* in the subgame following $(Steal, 0)$, then player 1's optimal action is $(Steal, 0)$, following which player 2 chooses *Split* and the payoffs are $(100, 0)$.
- If player 2 chooses *Steal* in the subgames following both $(Split, 50)$ and $(Steal, 0)$, then player 1 has no optimal action (as in the second case).

Thus the game has two subgame perfect equilibria, in both of which player 1 chooses $(Steal, 0)$ and player 2 chooses *Steal* in the subgame following $(Split, x)$ for any $x < 50$ and *Split* in the subgame following $(Steal, x)$ for all x. The outcome is that player 1 chooses *Steal* and offers no transfer, and player 2 chooses *Split*.

Presumably the reason why the player on the right did not offer a transfer of zero is much the same as the reason why proposers in the ultimatum game do not generally offer zero.