Economics 316

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Solutions to problems for Tutorial 10

1. (a) Firm 2's payoff function is

$$\begin{cases} 0 & \text{if } q_2 = 0 \\ q_2(P(q_1 + q_2) - c) - f & \text{if } q_2 > 0. \end{cases}$$

As before firm 2's best response to q_1 is $(\alpha - c - q_1)/2$ if firm 2's profit is nonnegative for this output; otherwise its best response is the output of zero. Firm 2's profit when it produces $(\alpha - c - q_1)/2$ and firm 1 produces q_1 is

$$\frac{\alpha - c - q_1}{2}\left(\alpha - c - \frac{\alpha - c - q_1}{2} - q_1\right) - f = \left(\frac{\alpha - c - q_1}{2}\right)^2 - f,$$

which is nonnegative if

$$\left(\frac{\alpha - c - q_1}{2}\right)^2 > f,$$

or if $q_1 \leq \alpha - c - 2\sqrt{f}$. Let $\overline{q} = \alpha - c - 2\sqrt{f}$. Then firm 2's best response function is

$$b_2(q_1) = \begin{cases} \frac{1}{2}(\alpha - c - q_1) & \text{if } q_1 < \overline{q} \\ \{0, \frac{1}{2}(\alpha - c - q_1)\} & \text{if } q_1 = \overline{q} \\ 0 & \text{if } q_1 > \overline{q}. \end{cases}$$

(If $q_1 = \overline{q}$ then firm 2's profit is zero whether it produces the output $\frac{1}{2}(\alpha - c - q_1)$ or the output 0; both outputs are optimal. Note that $\overline{q}$ if negative if $f > \frac{1}{4}(\alpha - c)^2$, in which case $b_2(q_1) = 0$ for all values of q_1.)

Thus firm 2's optimal strategy has a jump: for outputs of firm 1 slightly less than $\overline{q}$ firm 2 wants to produce a positive output (and earn a small profit), while for outputs of firm 1 slightly greater than $\overline{q}$ it wants to produce an output of zero.

(b) For the subgame perfect equilibrium there are three cases. In the first case, the output $\overline{q}$ at which firm 2's best response function jumps down to zero is greater than $\frac{1}{2}(\alpha - c)$. In the second case, we have $\overline{q} < \frac{1}{2}(\alpha - c)$. (It is helpful to refer to the figures in the slides.) Note that the condition $\overline{q} > \frac{1}{2}(\alpha - c)$ is equivalent to $f < \frac{1}{16}(\alpha - c)^2$.

$\overline{q} > \frac{1}{2}(\alpha - c)$ ($f < \frac{1}{16}(\alpha - c)^2$) In this case firm 1's optimal output is either $\overline{q}$ or $\frac{1}{2}(\alpha - c)$, depending on which output yields the higher profit when firm 2 responds optimally. (See the figure in the slides.) If firm 1's output is $\overline{q}$ then firm 2's output is 0 and firm 1's profit is

$$\overline{q}(\alpha - c - \overline{q}) = 2\sqrt{f}(\alpha - c - 2\sqrt{f}).$$

If firm 1's output is $\frac{1}{2}(\alpha - c)$ then firm 2's output is $\frac{1}{4}(\alpha - c)$ and firm 1's profit is

$$\tfrac{1}{2}(\alpha - c)(\alpha - c - \tfrac{3}{4}(\alpha - c)) = \tfrac{1}{8}(\alpha - c)^2.$$

Thus firm 2's optimal output is 0 if

$$2\sqrt{f}(\alpha - c - 2\sqrt{f}) > \tfrac{1}{8}(\alpha - c)^2.$$

The left-hand side of this inequality is a quadratic in $\sqrt{f}$ that is zero when $\sqrt{f}$ is zero or $\sqrt{f} = \frac{1}{2}(\alpha - c)$, or when $f = 0$ or $f = \frac{1}{4}(\alpha - c)^2$. Now, for $\overline{q} > \frac{1}{2}(\alpha - c)$ (which is required in this case), we need $f < \frac{1}{16}(\alpha - c)^2$. In the range from 0 to $\frac{1}{16}(\alpha - c)^2$, the function on the left-hand side is therefore increasing, so that the inequality is satisfied if

$$\sqrt{f} > \tfrac{2-\sqrt{2}}{8}(\alpha - c)$$

or

$$f > \tfrac{3-2\sqrt{2}}{32}(\alpha - c)^2.$$

Taking into account the condition for $\overline{q} > \frac{1}{2}(\alpha - c)$, we conclude that firm 2's output is zero in a subgame perfect equilibrium if

$$\tfrac{3-2\sqrt{2}}{32}(\alpha - c)^2 < f < \tfrac{1}{16}(\alpha - c)^2.$$

$\overline{q} < \frac{1}{2}(\alpha - c)$ **(**$f > \frac{1}{16}(\alpha - c)^2$**)** In this case firm 1's optimal output, given firm 2's response, is $\frac{1}{2}(\alpha - c)$, in which case firm 2's output is zero.

We conclude that the condition on f for firm 2's output in a subgame perfect equilibrium to be zero is

$$f \geq \frac{3-2\sqrt{2}}{32}(\alpha - c)^2.$$

2. (a) The situation is modeled by the following extensive game.

Players The parent and the child.

Histories $\varnothing$, the set of actions a of the child, and the set of sequences (a, t), where a is an action of the child and t is a transfer from the parent to the child.

Player function $P(\varnothing)$ is the child, $P(a)$ is the parent for every value of a.

Preferences The child's preferences are represented by the payoff function $c(a) + t$ and the parent's preferences are represented by the payoff function $\min\{p(a) - t, c(a) + t\}$.

(b) To find the subgame perfect equilibria of this game, first consider the parent's optimal actions in the subgames of length 1. Consider the subgame following the choice of a by the child. We have $p(a) > c(a)$ (by assumption), so if the parent makes no transfer her payoff is $c(a)$. If she transfers \$1 to the child then her payoff increases to $c(a) + 1$. As she increases the transfer her payoff increases until $p(a) - t = c(a) + t$; that is, until $t = \frac{1}{2}(p(a) - c(a))$. (If she increases the transfer any more, she has less money than her child.) Thus the parent's optimal action in the subgame following the choice of a by the child is $t = \frac{1}{2}(p(a) - c(a))$.

Now consider the whole game. Given the parent's optimal action in each subgame, a child who chooses a receives the payoff $c(a) + \frac{1}{2}(p(a) - c(a)) = \frac{1}{2}(p(a) + c(a))$. Thus in a subgame perfect equilibrium the child chooses the action that maximizes $p(a) + c(a)$, the sum of her own private income and her parent's income.

Source: Becker, Gary S. (1974), "A theory of social interactions", *Journal of Political Economy* **82**, 1063–1093.