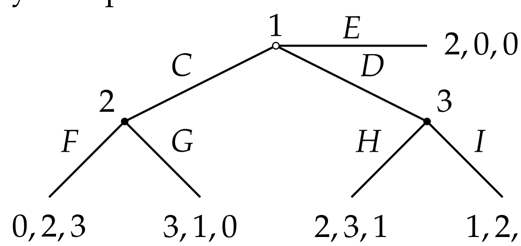
Economics 316

Fall 2017 Martin J. Osborne

Solutions to Problem Set 9

1. (a) The game may be represented as follows.



   (b) Players: $\{1, 2\}$. Terminal histories: $\{(C, E, G), (C, E, H), (C, F), D\}$. Player function: $P(\varnothing) = 1$, $P(C) = 2$, $P(C, E) = 1$.

2. (a) Player 1 moves at the start of the game, when she has 3 actions, and after all six histories of length two, after which she has 2 actions. Thus she has $3 \times 2^6 = 192$ strategies.

   (b) Player 2 moves after each of the three histories of length 1. In each case she has two actions. Thus she has $2^3 = 8$ strategies.

3. Player 1 has 9 moves at the start of the game, then 7 moves after each of the 9×8 histories at which she moves on her next turn, 5 after each of the $9 \times 8 \times 7 \times 6$ histories at which she moves on her next turn, and 3 after each of the $9 \times 8 \times 7 \times 6 \times 5 \times 4$ histories at which she moves on her next turn. Thus her number of strategies is

$$9 \times 7^{(9\times 8)} \times 5^{(9\times 8\times 7\times 6)} \times 3^{(9\times 8\times 7\times 6\times 5\times 4)}.$$

   Using the fact that $x^k = 10^{k \log_{10} x}$, this number is approximately $10^{31,031}$.

   Similarly, player 2 has

$$8^9 \times 6^{(9\times 8\times 7)} \times 4^{(9\times 8\times 7\times 6\times 5)} \times 2^{(9\times 8\times 7\times 6\times 5\times 4\times 3)}$$

   strategies, or approximately $10^{64,122}$.

   By contrast, the number of atoms in the observable universe is estimated to be around 10^{80}.

4. (a) The strategic form of the game follows.

	C	D
AE	$1,4$	$3,3$
AF	$0,0$	$3,3$
BE	$2,2$	$2,2$
BF	$2,2$	$2,2$

(b) The Nash equilibria of the game are (AF, D), (BE, C), and (BF, C).

(c) The subgame perfect equilibrium of the game is (BE, C).

(d) The players' payoffs in the Nash equilibria are $(2,2)$ and $(3,3)$; their payoffs in the subgame perfect equilibrium are $(2,2)$. That is, the subgame perfect equilibrium is Pareto dominated by one of the Nash equilibria.

5. (a) The following extensive game models the situation.

Players The firm and the union.

Histories $\varnothing$ and all sequences of the form w, (w, Y), (w, Y, L) and (w, N) for nonnegative numbers w and L (where w is a wage, Y means accept, N means reject, and L is the number of workers hired).

Player function $P(\varnothing)$ is the union, and, for any nonnegative number w, $P(w)$ and $P(w, Y)$ are the firm.

Preferences The firm's preferences are represented by its profit, and the union's preferences are represented by the value of wL (which is zero after any history (w, N)).

The game is illustrated in the Figure 1, where $\pi(w, L)$ denotes the firm's profit:

$$\pi(w, L) = \begin{cases} L(100 - L) - wL & \text{if } L \leq 50 \\ 2500 - wL & \text{if } L > 50. \end{cases}$$

Note that, as for any extensive game in which a player has a continuum of available actions after some history, this representation is not complete. At each of the histories after which the player who is moving has a continuum of actions, only one action is indicated.

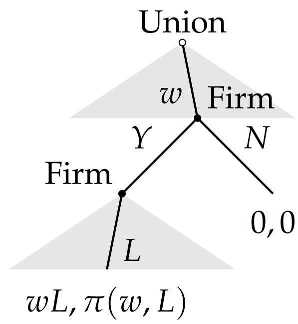


Figure 1. The game in Problem 5.

(b) First consider the subgame following a history (w, Y), in which the firm accepts the wage demand w. In a subgame perfect equilibrium, in this subgame the firm chooses L to maximize its profit, given w. Its profit is

$$\pi(w, L) = \begin{cases} L(100 - w - L) & \text{if } L \leq 50 \\ 2500 - wL & \text{if } L > 50. \end{cases}$$

Thus for $L \leq 50$ the profit is a quadratic function of L that is zero when $L = 0$ and $L = 100 - w$ and reaches a maximum in between. For $L > 50$ the profit is decreasing in L. An example (for $w = 20$) is shown in Figure 2.

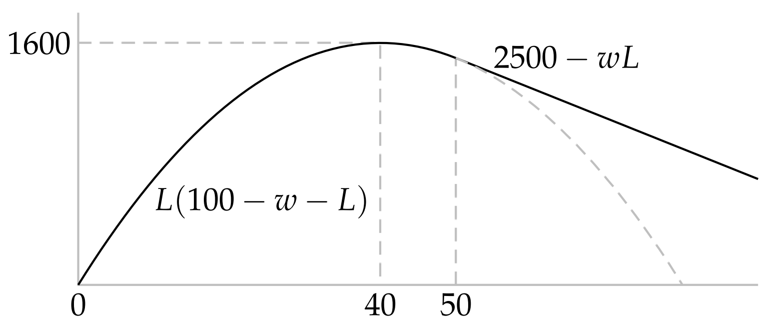


Figure 2. The profit function of the firm Problem 5 for $w = 20$.

Thus the value of L that maximizes the firm's profit is

$$\begin{cases} \frac{1}{2}(100 - w) & \text{if } w \leq 100 \\ 0 & \text{if } w > 100. \end{cases}$$

Given the firm's optimal action in such a subgame, consider the subgame following a history w, in which the firm has to decide whether to accept or reject w. For any w the firm's profit, given its subsequent optimal choice of L, is nonnegative; if $w < 100$ this profit is positive, while if $w \geq 100$ it is 0. Thus in a subgame perfect equilibrium, the firm accepts any demand $w < 100$ and either accepts or rejects any demand $w \geq 100$.

Finally consider the union's choice at the beginning of the game. If it chooses $w < 100$ then the firm accepts and chooses $L = (100 - w)/2$, yielding the union a payoff of $w(100 - w)/2$. If it chooses $w > 100$ then the firm either accepts and chooses $L = 0$ or rejects; in both cases the union's payoff is 0. Thus the best value of w for the union is the number that maximizes $w(100 - w)/2$. This function is a quadratic that is zero when $w = 0$ and when $w = 100$ and reaches a maximum in between; thus its maximizer is $w = 50$.

In summary, in a subgame perfect equilibrium the union's strategy is $w = 50$, and the firm's strategy accepts any demand $w < 100$ and chooses $L = (100 - w)/2$, and either rejects a demand $w \geq 100$ or accepts such a demand and chooses $L = 0$. The outcome of any equilibrium is that the union demands $w = 50$ and the firm chooses $L = 25$.

(c) Yes. In any subgame perfect equilibrium the union's payoff is $(50)(25) = 1250$ and the firm's payoff is $(25)(75) - (50)(25) = 625$. Thus both parties are better off at the outcome (w, L) than they are in the unique subgame perfect equilibrium if and only if $L \leq 50$ and

$$\begin{aligned} wL &> 1250 \\ L(100 - L) - wL &> 625 \end{aligned}$$

or $L \geq 50$ and

$$\begin{aligned} wL &> 1250 \\ 2500 - wL &> 625. \end{aligned}$$

These conditions are satisfied for a nonempty set of pairs (w, L). For example, if $L = 50$ the conditions are satisfied by $25 < w < 37.5$; if $L = 100$ they are satisfied by $12.5 < w < 18.75$.

(d) There are many Nash equilibria in which the firm "threatens" to reject high wage demands. In one such Nash equilibrium the firm threatens to reject any positive wage demand. In this equilibrium the union's strategy is $w = 0$, and the firm's strategy rejects any demand $w > 0$, and accepts the demand $w = 0$ and chooses $L = 50$. (The union's payoff is 0 no matter what demand it makes; given $w = 0$, the firm's optimal action is $L = 50$.)

6. I claim that every subgame perfect equilibrium results in the first candidate's entering at $\frac{1}{2}$, the second candidate's staying out, and the third candidate's entering at $\frac{1}{2}$.

   To show this, first consider the best response of candidate 3 to each possible pair of actions of candidates 1 and 2. Figure 3 illustrates these optimal actions in every case that candidate 1 enters. (If candidate 1 does not enter then the subgame is exactly the two-candidate game.)

   Now consider the optimal action of candidate 2, given x_1 and the outcome of candidate 3's best response, as given in Figure 3. In the figure, take a value of x_1 and look at the outcomes as x_2 varies; find the value of x_2 that induces the best outcome for candidate 2. For example, for $x_1 = 0$ the only value of x_2 for which candidate 2 does not lose is $\frac{2}{3}$, at which point she ties with the other two candidates. Thus when candidate 1's strategy is $x_1 = 0$, candidate 2's best action, given candidate 3's best response, is $x_2 = \frac{2}{3}$, which leads to a three-way tie. We find that the outcome of the optimal value of x_2, for each value of x_1, is given as follows.

$$\begin{cases} 1, 2, \text{and } 3 \text{ tie } (x_2 = \frac{2}{3}) & \text{if } x_1 = 0 \\ 2 \text{ wins} & \text{if } 0 < x_1 < \frac{1}{2} \\ 1 \text{ and } 3 \text{ tie } (2 \text{ stays out}) & \text{if } x_1 = \frac{1}{2} \\ 2 \text{ wins} & \text{if } \frac{1}{2} < x_1 < 1 \\ 1, 2, \text{and } 3 \text{ tie } (x_2 = \frac{1}{3}) & \text{if } x_1 = 1. \end{cases}$$

   Finally, consider candidate 1's best strategy, given the responses of candidates 2 and 3. If she stays out then candidates 2 and 3 enter at m and tie. If she enters then the best position at which to do so is $x_1 = \frac{1}{2}$, where she ties with candidate 3. (For every other position she either loses or ties with both of the other candidates.)

   We conclude that in every subgame perfect equilibrium the outcome is that candidate 1 enters at $\frac{1}{2}$, candidate 2 stays out, and candidate 3 enters at $\frac{1}{2}$. (There are many subgame perfect equilibria, because after many histories candidate 3's optimal action is not unique.)

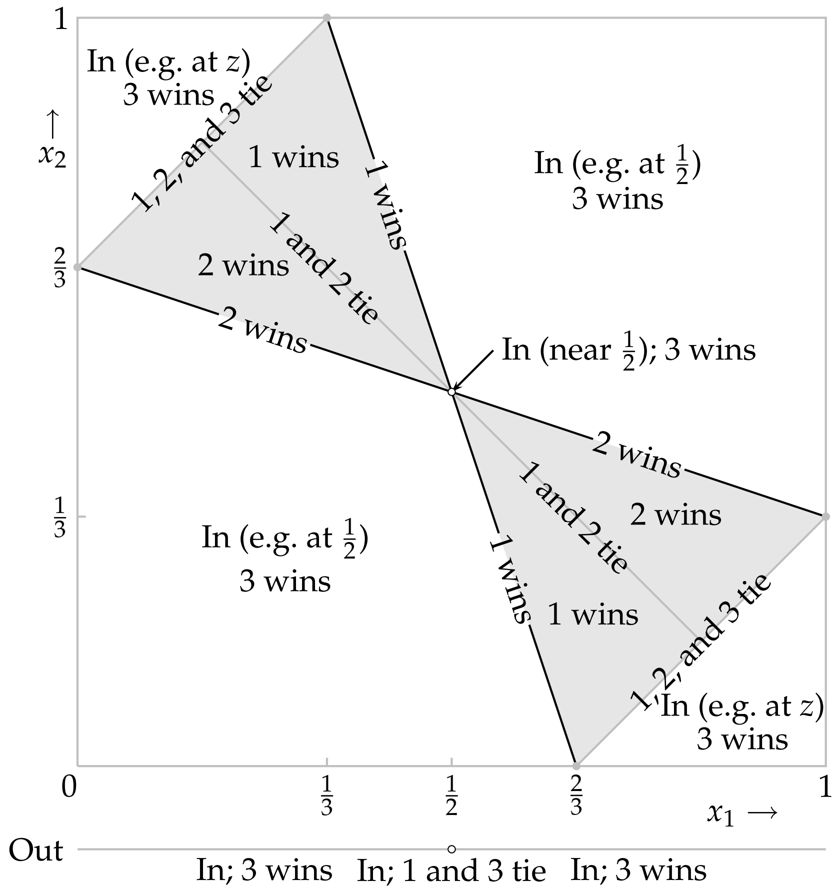


Figure 3. The outcome of a best response of candidate 3 to each pair of actions by candidates 1 and 2. The best response for any point in the gray shaded area (including the black boundaries of this area, but excluding the other boundaries) is *Out*. The outcome at each of the four small disks at the outer corners of the shaded area is that all three candidates tie. The value of z is $1 - \frac{1}{2}(x_1 + x_2)$.