UNIVERSITY OF TORONTO
Faculty of Arts and Science

ECO 316 (Applied Game Theory)
Instructor: Martin J. Osborne

Midterm Exam
20 October 2017

Duration: 1 hour 50 minutes

No aids allowed

The number in brackets at the start of each question is the number of points the question is worth.

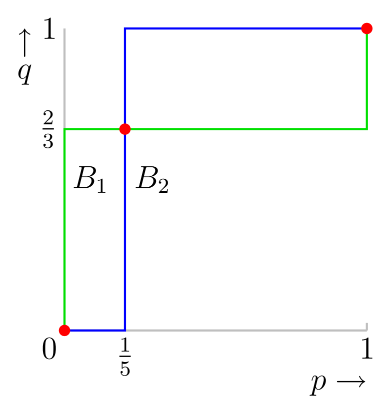
Answer all questions.

TO OBTAIN CREDIT, YOU MUST GIVE ARGUMENTS TO SUPPORT YOUR ANSWERS.

1. (a) [8] Find all mixed strategy Nash equilibria of the following strategic game. (Remember that in a mixed strategy Nash equilibrium a player may assign probability 1 to a single action.)

	L	R
T	$2, 0$	$2, 4$
B	$3, 1$	$0, 0$

Solution: The best response functions are shown in the following figure. The game has three mixed strategy Nash equilibria: $((0,1),(1,0))$, $((1,0),(0,1))$, and $((\frac{1}{5},\frac{4}{5}),(\frac{2}{3},\frac{1}{3}))$.



(b) [12] Does the following strategic game have a mixed strategy Nash equilibrium in which each player assigns positive probability *only* to her actions A and C?

	A	B	C
A	$2, 2$	$2, 4$	$6, 3$
B	$4, 2$	$6, 6$	$0, 3$
C	$3, 6$	$3, 1$	$3, 3$

Solution: For a game to have a mixed strategy Nash equilibrium in which a player assigns positive probability to only two of her actions, (a) the player's expected payoffs to these two actions must be the same *and* (b) her expected payoffs to any other actions must be no greater than her expected payoffs to the actions to which she assigns positive probability.

Consider a pair of mixed strategies for which each player assigns positive probability only to actions A and C.

Denote by q the probability that player 2 assigns to A, so that the probability she assigns to C is $1 - q$. For the expected payoffs of player 1 to the actions A and C to be the same, we need

$$2q + 6 - 6q = 3$$

or $q = \frac{3}{4}$. For this mixed strategy of player 2, player 1's expected payoffs to A and C are both 3, and her expected payoff to B is $4(\frac{3}{4}) = 3$.

Now denote by p the probability that player 1 assigns to A, so that the probability she assigns to C is $1 - p$. For the expected payoffs of player 1 to the actions A and C to be the same, we need

$$2q + 6 - 6q = 3$$

or $p = \frac{3}{4}$. For this mixed strategy of player 1, player 2's expected payoffs to A and C are 3 and her expected payoff to B is $4(\frac{3}{4}) + 1(\frac{1}{4}) > 3$, violating the requirement that this payoff be at most her expected payoff to A and C.

Thus the game has no mixed strategy Nash equilibrium in which each player assigns positive probability only to A and C.

(c) [5] In the following strategic game, is any action of either player strictly dominated by a mixed strategy?

	A	B	C
T	$1,5$	$3,6$	$2,5$
B	$5,3$	$2,1$	$0,2$

Solution: Notice first that for player 2, A weakly dominates C and that (a) if player 1 chooses T then player 2's payoff to B is higher than her common payoff to A and C, and (b) if player 1 chooses B then player 2's payoff to A exceeds her payoff to C. Thus if player 2 uses A with high probability and B with low probability, then whether player 1 chooses T or B, player 2's expected payoff is greater than her expected payoff if she uses C. That is, such a mixed strategy strictly dominates C.

Specifically, the action C of player 2 is strictly dominated by the mixed strategy that assigns probability $\frac{2}{3}$ to A and probability $\frac{1}{3}$ to B, because

- if player 1 chooses T then player 2's payoff to C is 5 and her expected payoff to the mixed strategy is $\frac{2}{3} \cdot 5 + \frac{1}{3} \cdot 6 > 5$
- if player 1 chooses B then player 2's payoff to C is 2 and her expected payoff to the mixed strategy is $\frac{2}{3} \cdot 3 + \frac{1}{3} \cdot 1 > 2$.

(In fact a mixed strategy that assigns any probability greater than $\frac{1}{2}$ to A and the remaining probability to B strictly dominates C.)

(d) [5] Consider the following strategic game. Which actions for each player are consistent with her being rational (in the sense defined in class), with her knowing that the other player is rational, with her knowing that the other player knows she is rational, and so on?

	A	B	C
A	$1,5$	$3,6$	$2,4$
B	$5,3$	$2,1$	$0,4$
C	$2,3$	$4,2$	$3,0$

Solution: An action is consistent with rationality if it is a best response to some belief, which is equivalent to its not be strictly dominated by a mixed strategy.
Now, A is strictly dominated by C for player 1, so choosing A is not rational. So if player 2 knows that player 1 is rational, she knows that player 1 will not choose A. Once A is eliminated for player 1, B is strictly dominated by A for player 2. So if player 2 knows player 1 is rational, choosing B is not rational for her.
In the resulting game, no action is strictly dominated. Thus the actions B and C for player 1 and the actions A and C for player 2 are consistent with each player's being rational and knowing that the other player is rational.

2. Each of two players chooses a number from the set $\{0, 1, 2, \ldots, M\}$, where $M \geq 3$. If the sum of the numbers chosen by the players is less than or equal to M, each player's payoff is the number she chose. Otherwise, each player's payoff is zero.

(a) [5] Is the action of choosing 0 strictly dominated for player 1?

Solution: No: If player 2 chooses M then player 1's payoff is zero if she chooses 0, and is also zero if she chooses any other action.

(b) [10] Find all the pure strategy Nash equilibria of this game.

Solution: Consider the best responses of player 1. If player 2's action is k with $k \leq M-1$, then player 1's best response is $M-k$. If player 2's action is M, then all actions of player 1 yield the payoff 0, so all actions of player 1 are best responses.
The roles of player 1 and player 2 in the game are symmetric, so player 2's best responses take the same form.
Thus every pair $(k, M-k)$ with k in $\{0, \ldots, M\}$ is a Nash equilibrium together with (M, M).

3. [20] Consider a variant of the example of Cournot's duopoly game considered in class in which the payoff of firm 1 is its profit if this profit is negative, but otherwise (that is, if firm 1's profit is nonnegative) is firm 1's *output* (that is, sales), and the payoff of firm 2 is always its profit (as assumed in class).

(Each firm's cost of producing q units of output is cq and the price when the total output of the firms is Q is $\alpha - Q$, with $\alpha > c$.)

Find the set of Nash equilibria of the game.

Solution: If firm 2's output is q_2, firm 1's profit when its output is q_1 is

$$q_1(\alpha - q_1 - q_2 - c).$$

Thus firm 1's profit is nonnegative if and only if $\alpha - q_1 - q_2 - c \geq 0$, or $q_1 \leq \alpha - q_2 - c$. Hence firm 1's payoff if its output is q_1 and firm 2's output is q_2 is

$$\begin{cases} q_1 & \text{if } q_1 \leq \alpha - q_2 - c \\ q_1(\alpha - q_1 - q_2 - c) & \text{if } q_1 > \alpha - q_2 - c. \end{cases}$$

Thus for any output $q_2 \leq \alpha - c$, firm 1's best response to q_2 is $\alpha - q_2 - c$. (For any higher output, its profit is negative. If $q_2 > \alpha - c$ then firm 1's profit is negative whenever its output is positive, so its best response is an output of zero.)

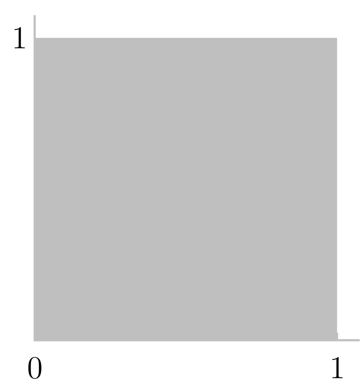
Firm 2's best response to any output q_1 is is $\frac{1}{2}(\alpha - q_1 - c)$ (as in the example discussed in class).

Thus an action pair (q_1, q_2) is a Nash equilibrium if

$$\begin{aligned} q_1 &= \alpha - q_2 - c \\ q_2 &= \tfrac{1}{2}(\alpha - q_1 - c). \end{aligned}$$

Thus the unique Nash equilibrium is $(\alpha - c, 0)$.

4. Consider Hotelling's model of electoral competition with *four* candidates. Assume that each candidate wants to maximize her probability of winning and that no candidate has the option of exiting the race. (If k candidates tie for first place, each of them wins the probability $1/k$.) Suppose that the distribution of the voters' favorite positions is uniform from 0 to 1. That is, the distribution looks like this:



(a) [7] Does the game have a Nash equilibrium in which the positions of all four candidates are equal to $\frac{1}{2}$?

Solution: No: If all four candidates choose the position $\frac{1}{2}$ then each of them wins with probability $\frac{1}{4}$. If any of them deviates to a position slightly less than $\frac{1}{2}$ or slightly greater than $\frac{1}{2}$, she wins with probability 1.

(b) [8] Does the game have a Nash equilibrium in which the positions of two candidates are equal to $\frac{1}{4}$ and the positions of the other two candidates are equal to $\frac{3}{4}$?

Solution: Yes: For this action profile, each candidate wins with probability $\frac{1}{4}$.
If a candidate at $\frac{1}{4}$ deviates to a position less than $\frac{1}{4}$, she loses because her vote share decreases to less than $\frac{1}{4}$, the vote share of the other candidate at $\frac{1}{4}$ increases to more than $\frac{1}{4}$, and the vote shares of the other two candidates do not change.
If a candidate at $\frac{1}{4}$ deviates to a position greater than $\frac{1}{4}$, she loses because her vote share decreases to less than $\frac{1}{4}$, the vote share of the other candidate at $\frac{1}{4}$ increases to more than $\frac{1}{4}$, and the vote shares of the other two candidates decrease.
Similarly any deviation by a candidate at $\frac{3}{4}$ causes her to lose rather than tie for first place.

5. [20] Consider a variant of the crime-reporting model in which each individual independently observes the crime with probability q, which may be less than 1 (instead of observing the crime for sure). Only individuals who observe the crime have the option to report it. Suppose that every individual reports the crime with probability p in the event she observes it. Then the probability that an individual reports the crime is qp (the probability that she observes it times the probability she reports it in the event she observes it).

As in the model discussed in class, an individual who reports the crime incurs the cost $c > 0$, and if the crime is reported then every individual obtains the benefit $v > c$. Denote the number of individuals by n.

Find a ("symmetric") mixed strategy equilibrium of the strategic game that models this situation in which the probability that each individual reports the crime, conditional on observing it, is the same.

Solution: Notice that in this variant of the model, the probability an individual reports the crime is qp, rather than p as in the original model. Thus the calculations are the same as the calculations for the original model, except that p is replaced by qp.

If everyone else who observes the crime reports it with probability p, then an individual who observes it gets the payoff $v - c$ if she reports it. If she doesn't report it, the probability that at least one other player reports it is $1-(1-qp)^{n-1}$, so her expected payoff is

$$v(1-(1-qp)^{n-1}).$$

For an equilibrium in which $0 < p < 1$, we need these two payoffs to be equal, or

$$v - c = v(1-(1-qp)^{n-1})$$

or

$$c/v = (1 - qp)^{n-1}$$

or

$$p = (1 - (c/v)^{1/(n-1)})/q.$$

Up to this point, differ from the calculations for the original model only in that qp replaces p. Now there is a wrinkle: we have to check whether the value of p lies between 0 and 1. In the original model, it always does so. But here, it may not. It is always positive, but is less than 1 only if

$$q > 1 - (c/v)^{1/(n-1)}.$$

If this inequality is not satisfied, then the expected payoff to reporting a signal exceeds the expected payoff to not reporting it even for $p = 1$, so there is a Nash equilibrium in which every individual who gets a signal always reports it.

In summary, if

$$q > 1 - (c/v)^{1/(n-1)}$$

then the game has a Nash equilibrium in which every individual who gets a signal reports it with probability

$$p = (1 - (c/v)^{1/(n-1)})/q$$

and if

$$q \leq 1 - (c/v)^{1/(n-1)}$$

then the game has a Nash equilibrium in which every individual who gets a signal reports it with probability 1.