ECO316: Applied game theory

Lecture 9

Martin J. Osborne

Department of Economics
University of Toronto

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Table of contents

Common value auctions

- In many auctions, bidders' valuations are not independent
- Instead, bidders' valuations may be related to each other
- Even a buyer of a work of art may care about its resale value, which depends on other people's valuations of it
- Interdependence of values introduces considerations not present when values are independent

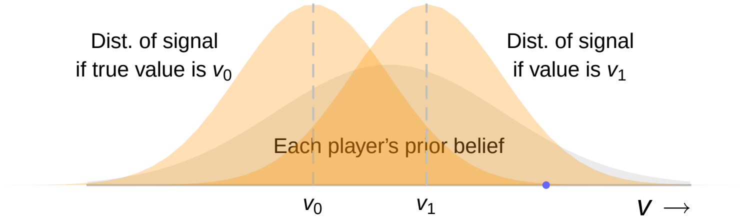
Common value auctions

Drilling for oil

- All firms value oil in the same way
- But no firm knows amount available
- Each firm i privately takes a sample, which generates a signal s_i about amount available
- Samples differ, so firms' estimates of amount available differ
- *If* firm i were to know all firms' signals, $(s_1, \ldots, s_n)$, then its estimate of the amount available would be $V(s_1, \ldots, s_n)$

Common value auctions

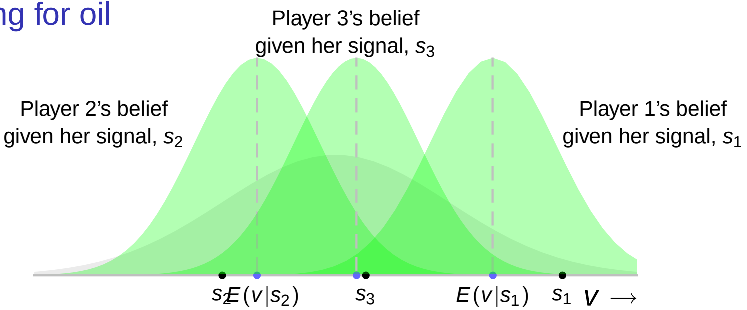
Drilling for oil



- Each player sees only her *own* signal
- On basis of her signal and prior belief, each player can calculate probabilistic estimate of value (using Bayes' law)
- Different players get different signals, so their estimates of the value based on these signals differ

Common value auctions

Drilling for oil



- Each player sees only her *own* signal
- On basis of her signal and prior belief, each player can calculate probabilistic estimate of value (using Bayes' law)
- Different players get different signals, so their estimates of the value based on these signals differ

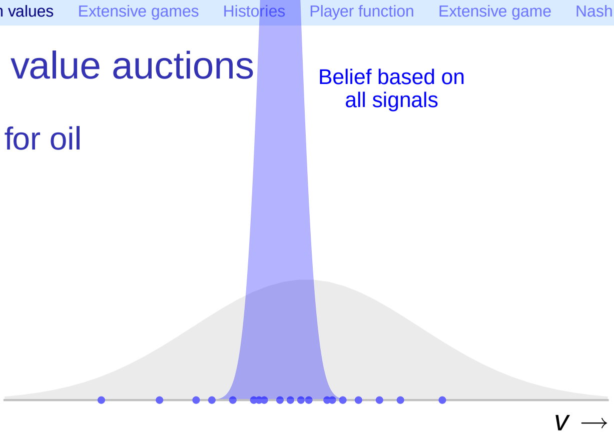
Common value auctions

Drilling for oil

- Each black dot represents the signal received by a player
- Each blue dot represents the expectation of v given the corresponding signal—that is, $E(v \mid \text{signal is } s_i)$

Common value auctions

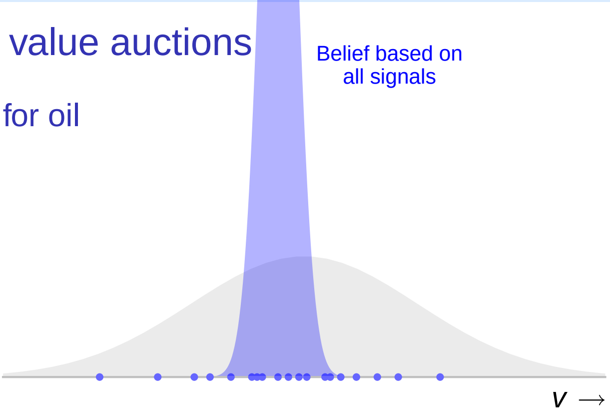
Drilling for oil



- Consider second-price auction
- If every player's bid is the expectation of the value based solely on her own signal, the second highest bid will typically far exceed the actual value, and the winner's payoff will be negative
- Effect is known as winner's curse

Common value auctions

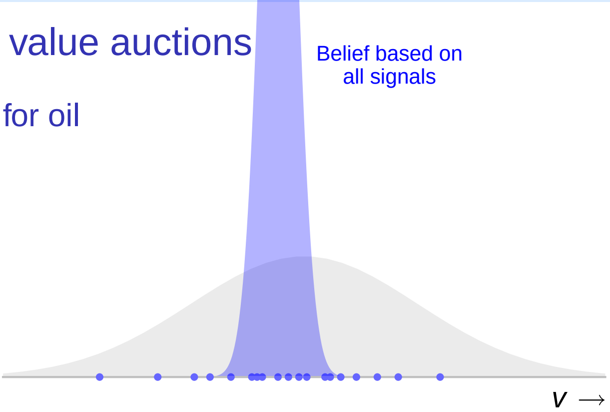
Drilling for oil



- Player know that if she wins, all other players' signals are lower than hers
- She should taken this information into account, and base her bid on estimate of value *conditional on winning* (given other players' strategies)

Common value auctions

Drilling for oil



- In Nash equilibrium of second-price auction, player i with signal s_i bids

  $E(v \mid i\text{'s signal is } s_i \textit{ and } \text{signals of all other players are} \leq s_i)$

- This expectation is typically much less than $E(v \mid s_i)$

Common value auctions

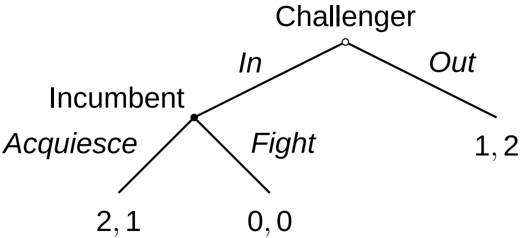
Summary

- Even for second-price rule, not an equilibrium in common value auction for each player to bid her valuation based on her own information
- Player who does so suffers “winner’s curse”
- Instead, player should base her bid on her estimate of the value of the object over all cases in which all other bids are lower than hers, given the other players’ equilibrium strategies

Extensive games with perfect information

- Strategic game is not natural model of situation in which actions are chosen sequentially
- Actions in strategic game *can* capture behavior that will unfold over time, but strategic game does not allow reevaluation of choices
- Model that explicitly captures sequential choices: *extensive game*

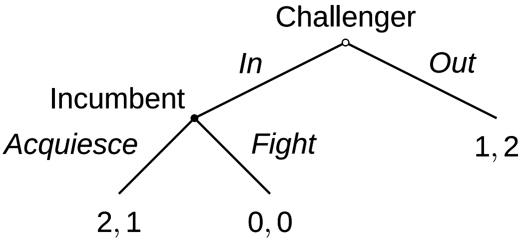
Example: entry game



- Two players, Challenger and Incumbent
- Small circle denotes start of game
- First, Challenger chooses *In* or *Out*
- If Challenger chooses *In*, Incumbent chooses *Acquiesce* or *Fight*
- If Challenger chooses *Out*, game ends
- Payoffs are numbers at bottom (challenger's payoff first)

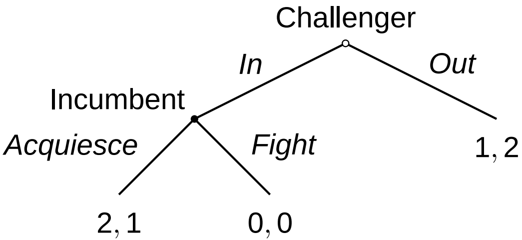
Histories

- A history is a sequence of actions beginning at start of game
- Example: in entry game, histories are $\varnothing$ (the start of the game), *In*, *Out*, (*In*, *Acquiesce*), and (*In*, *Fight*)



Terminal histories

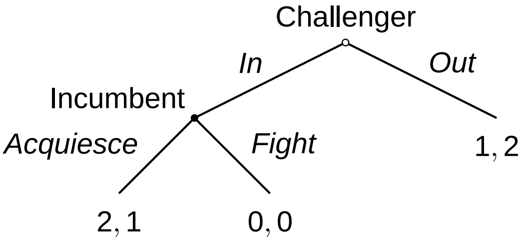
- A terminal history is a history that reaches end of game
- Example: in entry game, terminal histories are *Out*, (*In*, *Acquiesce*), and (*In*, *Fight*)



Player function

- Specifies player who moves after any nonterminal history
- Example: for entry game,

$$P(\varnothing) = \text{Challenger} \quad (\varnothing = \text{null history, start of game})$$
$$P(\mathit{In}) = \text{Incumbent}$$

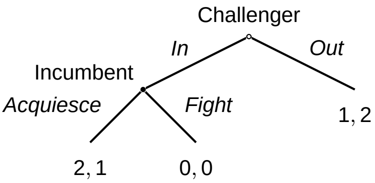


Extensive games with perfect information

An extensive game with perfect information consists of

For entry game:

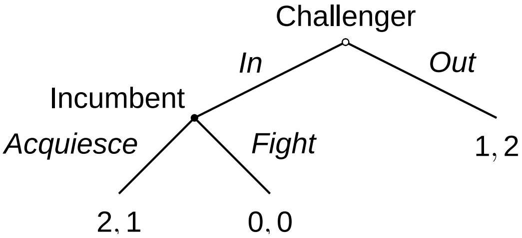
- set of players — $\{\text{Challenger}, \text{Incumbent}\}$
- set of terminal histories — $(In, Acquiesce)$, $(In, Fight)$, Out
- player function — $P(\varnothing) = \text{Challenger}$, $P(In) = \text{Incumbent}$
- players' payoffs for each terminal history — $(In, Acquiesce)$: $(2, 1)$; $(In, Fight)$: $(0, 0)$; Out: $(1, 2)$



Actions

- Actions available to players defined implicitly by terminal histories
- $A(h)$ = set of actions of player $P(h)$, who moves after history h
- For entry game,

$$A(\varnothing) = \{In, Out\}$$
$$A(In) = \{Acquiesce, Fight\}$$



Strategies

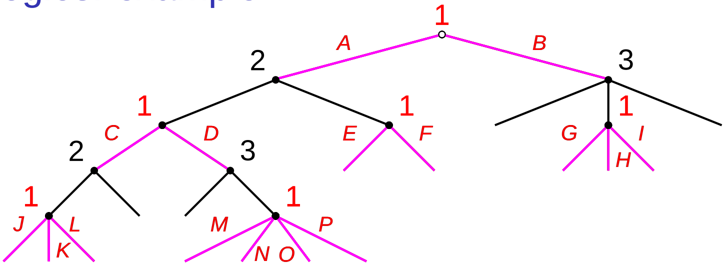
Definition

A strategy of player i in an extensive game with perfect information is a function that assigns an action in $A(h)$ to **EVERY** nonterminal history h for which $P(h) = i$.

Player's strategies may be found as follows:

- make list of *all* histories after which player moves
- *one* strategy is obtained by choosing, for each history, one of the player's actions available after that history
- player's *set* of strategies is found by taking all possible combinations of actions after the histories

Strategies: example

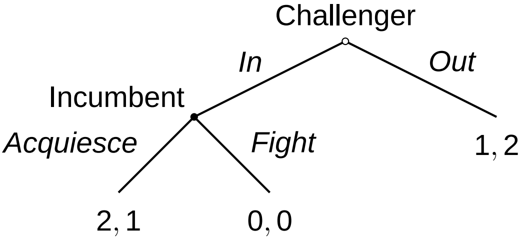


- Strategies of player 1:
  ACEGJM, *ACEGJN*, *ACEGJO*, *ACEGJP*, *ACEGKM*, *ACEGKN*, *ACEGKO*, *ACEGKP*, ... (many more)
- How many?

$$2 \times 2 \times 2 \times 3 \times 3 \times 4 = 288$$

- For any game: If player moves after k histories and has m_1 actions after one history, m_2 actions after another history, $\ldots$, m_k actions after kth history, total number of her strategies is $m_1 m_2 \ldots m_k$.

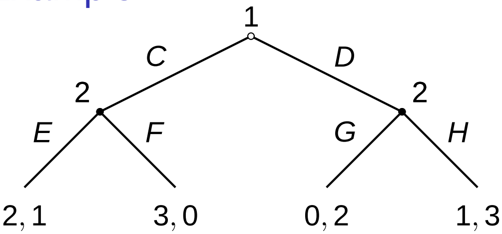
Strategies: Example



Challenger Moves only after null history. Two actions after this history, so two strategies: *In*, *Out*.

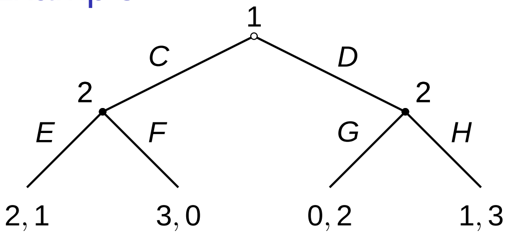
Incumbent Moves only after history *In*. Two actions after this history, so two strategies: *Acquiesce*, *Fight*.

Strategies: Example



Player 1 Moves only after null history. Two actions after this history, so two strategies: C, D.

Strategies: Example



Player 2 Moves after two histories:

C: two actions, E and F

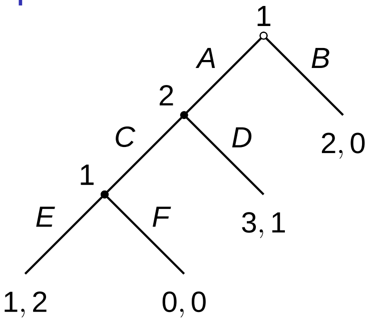
D: two actions, G and H

Hence four strategies:

- $s_2(C) = E$ and $s_2(D) = G$ (*EG* for short)
- $s_2(C) = E$ and $s_2(D) = H$ (*EH* for short)
- $s_2(C) = F$ and $s_2(D) = G$ (*FG* for short)
- $s_2(C) = F$ and $s_2(D) = H$ (*FH* for short)

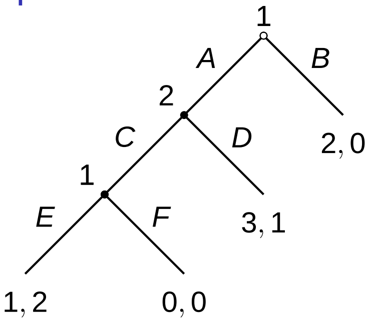
Strategy of player 2 in this game is *plan of action*.

Strategies: Example



Player 2 Moves after *one* history, *A*, and has 2 actions, *C* and *D*, so 2 strategies: *C*, *D*

Strategies: Example

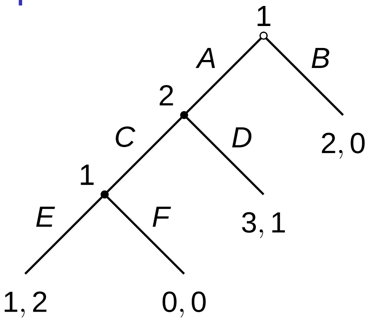


Player 1 Moves after

- null history: 2 actions, A and B
- history (A, C): 2 actions, E and F

So 4 strategies: AE, AF, BE, BF

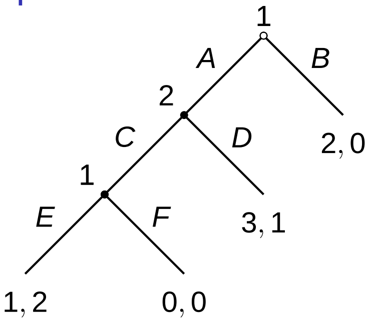
Strategies: Example



Note

- Each strategy of player 1 specifies action after history (A, C) *even if it specifies B at beginning of game*!
- In general: definition of strategy requires action to be specified for *every* history after which it is player's turn to move, *even histories not reached if strategy is followed*

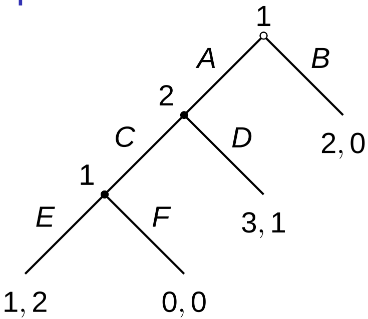
Strategies: Example



One interpretation of strategy *BE* of player 1:

1. Action *E* models behavior of player 1 if, by chance, she doesn't choose *B* at start of game (though she intends to)

Strategies: Example



Another interpretation of strategy BE of player 1:

2. When choosing between A and B,
   - player 1 has to think about action player 2 intends to take
   - player 1 knows that player 2's action depends on action player 2 thinks player 1 will take after history (A, C)

   Component E of player 1's strategy is her belief about player 2's belief about player 1's action after history (A, C)

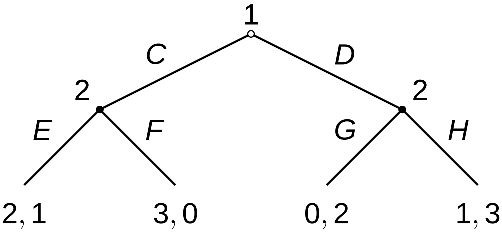
Strategic form of extensive game

Given any extensive game, can now define strategic game

- Players: players in extensive game
- Actions of player i: *strategies* of player i in extensive game
- Players' payoffs to action profile: payoffs to terminal history that results when the players follow their strategies

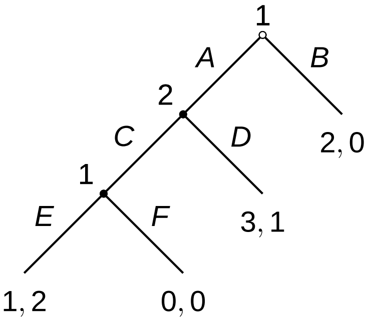
Resulting strategic game is strategic form of extensive game

Example of strategic form



	EG	*EH*	*FG*	*FH*
C	2, 1	2, 1	3, 0	3, 0
D	0, 2	1, 3	0, 2	1, 3

Example of strategic form



	C	D
AE	1, 2	3, 1
AF	0, 0	3, 1
BE	2, 0	2, 0
BF	2, 0	2, 0

Note duplicate strategies of player 1

Reduced strategic form:

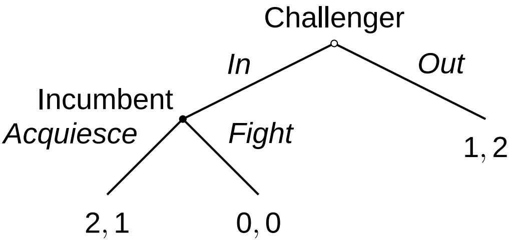
	C	D
AE	1, 2	3, 1
AF	0, 0	3, 1
X	2, 0	2, 0

Nash equilibrium

Definition

A Nash equilibrium of an extensive game with perfect information is a Nash equilibrium of its strategic form

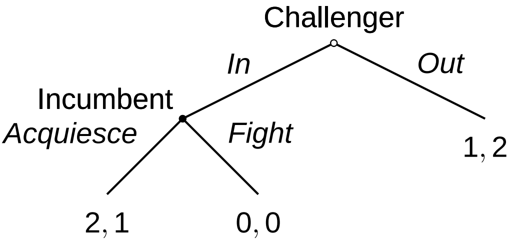
Example



	Acquiesce	*Fight*
In	2, 1	0, 0
Out	1, 2	1, 2

Nash equilibria: $(\mathit{In}, \mathit{Acquiesce})$ and $(\mathit{Out}, \mathit{Fight})$

Nash equilibrium: Example



Nash equilibria

$(In, Acquiesce)$ Both actions played in equilibrium; each is optimal when played

$(Out, Fight)$ *Out* played in equilibrium, but *Fight* not played

- *Fight* optimal given player 1 chooses *Out* (action of player 2 doesn't affect outcome)
- But *Fight* not optimal if history *In* occurs
  - *Fight* can be interpreted as *non-credible threat*

Nash equilibrium

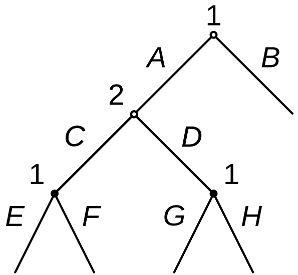
- In Nash equilibrium, each player's strategy optimal given other players' strategies
  $\Rightarrow$ each player's strategy optimal at *start* of game
- But a player's Nash equilibrium strategy may not be optimal after the game has started
- Notion of *subgame perfect equilibrium* requires that each player's strategy be optimal after *every* history, even histories that do not occur if every player follows her strategy

Subgames

For any nonterminal history h, subgame following h is part of game remaining once h has occurred

$\Rightarrow$ number of subgames $=$ number of nonterminal histories

Example



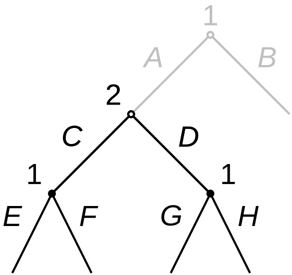
Subgame following $\varnothing$ (whole game)

Subgames

For any nonterminal history h, subgame following h is part of game remaining once h has occurred

$\Rightarrow$ number of subgames $=$ number of nonterminal histories

Example



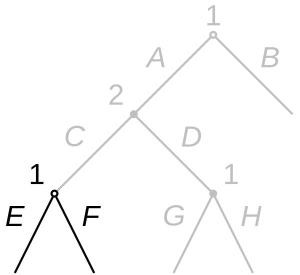
Subgame following A

Subgames

For any nonterminal history h, subgame following h is part of game remaining once h has occurred

$\Rightarrow$ number of subgames $=$ number of nonterminal histories

Example



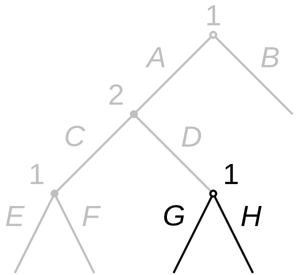
Subgame following (A, C)

Subgames

For any nonterminal history h, subgame following h is part of game remaining once h has occurred

$\Rightarrow$ number of subgames $=$ number of nonterminal histories

Example



Subgame following (A, D)

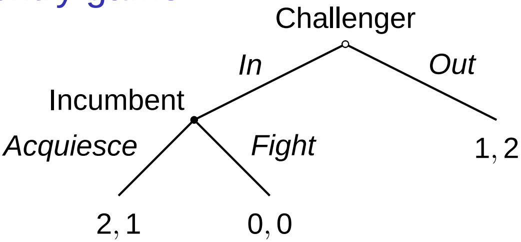
Subgame Perfect Equilibrium

Definition

A subgame perfect equilibrium of an extensive game with perfect information is a strategy profile s^* such that the strategy s_i^* of every player i is optimal, given the other players' strategies, in every subgame in which player i moves.

- In a Nash equilibrium, each player's strategy is optimal at start of game, so every subgame perfect equilibrium is a Nash equilibrium
- But not every Nash equilibrium is a subgame perfect equilibrium

Example: entry game



(*In*, *Acquiesce*)

- *In* optimal at start of game, given Incumbent's strategy
- *Acquiesce* optimal in subgame following *In*

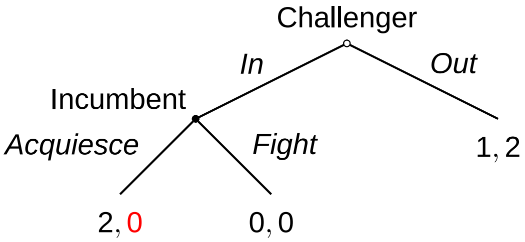
So subgame perfect equilibrium

(*Out*, *Fight*)

- *Out* optimal at start of game
- But *Fight* not optimal in subgame following *In*

So not subgame perfect equilibrium

Example: variant of entry game



$(In, Acquiesce)$

- *In* optimal at start of game, given Incumbent's strategy
- *Acquiesce* optimal in subgame following *In*

Subgame perfect equilibrium

$(Out, Fight)$

- *Out* optimal at start of game, given Incumbent's strategy
- *Fight* optimal after history *In*

Subgame perfect equilibrium

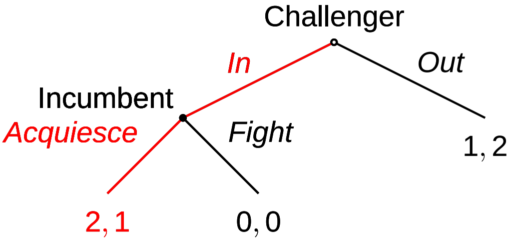
Finding subgame perfect equilibria

- If all terminal histories in game are finite, game has finite horizon
- In finite horizon game, subgame perfect equilibria can be found by backward induction

Backward induction

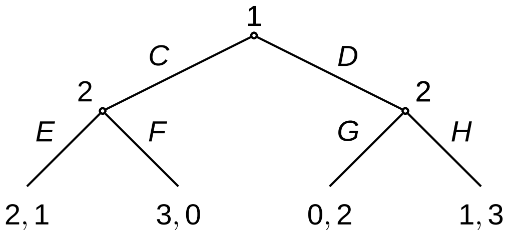
- Start by finding optimal action in every subgame of length one (at “end” of game)
- Given optimal actions in subgames of length one, find optimal action in each subgame of length two
- Continue to work backwards to start of game
- Strategy profiles found are subgame perfect equilibria

Example: entry game



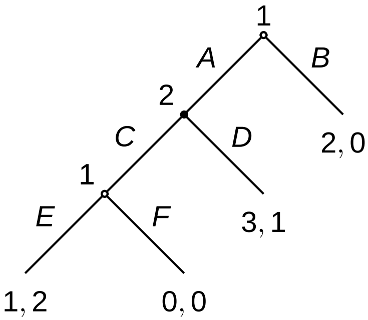
- One subgame of length 1, following history *In*: optimal action (of Incumbent) is *Acquiesce*
- One subgame of length 2 (whole game): optimal action (of Challenger), given outcome in subgame of length 1, is *In*
- Thus game has unique subgame perfect equilibrium, $(In, Acquiesce)$

Example



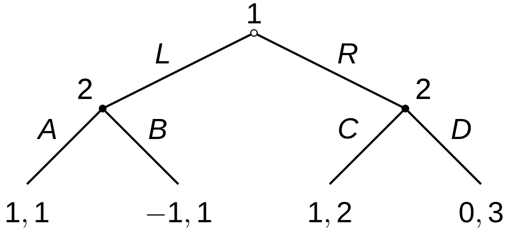
- Subgames of length one:
  - following C: E is optimal
  - following D: H is optimal
- Subgame of length two (whole game):
  - C is optimal
- Thus unique subgame perfect equilibrium: (C, EH)

Example



- Subgame of length 1, following (A, C): E is optimal
- Subgame of length 2, following A: C is optimal
- Subgame of length 3 (whole game): B is optimal
- Thus unique subgame perfect equilibrium: (BE, C)

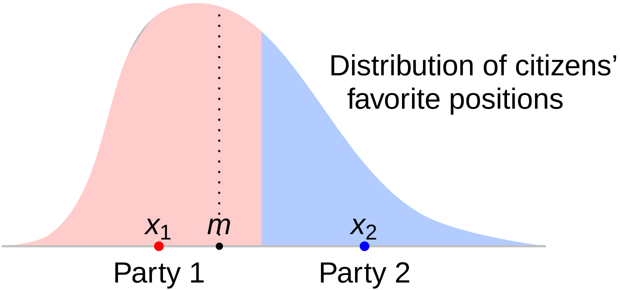
Example: game with indifference between outcomes



- Subgames of length one:
  - following L: A and B are both optimal
  - following R: D is optimal
- Subgame of length two (whole game): Need to consider separately each collection of optimal actions in subgames of length one:
  - AD: L is optimal
  - BD: R is optimal
- Thus *two* subgame perfect equilibria:
  - (L, AD)
  - (R, BD)

Example: electoral competition

- Variant of Hotelling's model, with sequential choice
- Two parties
  - party 1 chooses a position
  - party 2 observes party 1's position
  - party 2 chooses a position
- Each citizen votes for closest party, determining winner



Example: electoral competition

Extensive game

- Players: Two parties
- Terminal histories: all sequences (x_1, x_2), where x_i is a position for party i
- Player function:

$$P(\varnothing) = 1 \quad \text{(party 1 moves first)}$$

$$P(x_1) = 2 \text{ for all } x_1 \quad \text{(after any action of party 1, party 2 moves)}$$

- Preferences: Outcome of (x_1, x_2) determined by voters; each party's preferences over outcomes:

$$\text{win} \succ \text{tie} \succ \text{lose}$$

Note: number of histories is infinite, so game cannot be represented in diagram like one for entry game

Example: electoral competition

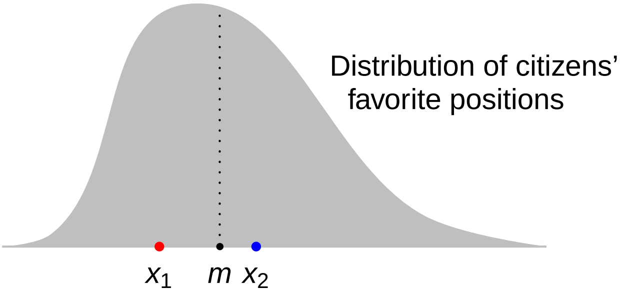
Subgame perfect equilibrium

- Use backward induction
- First consider subgames of length one: find best position for party 2, given any position for party 1
- Then consider whole game: find best position for party 1, given party 2's optimal actions in subgames

Example: electoral competition

Subgame perfect equilibrium: subgames of length 1

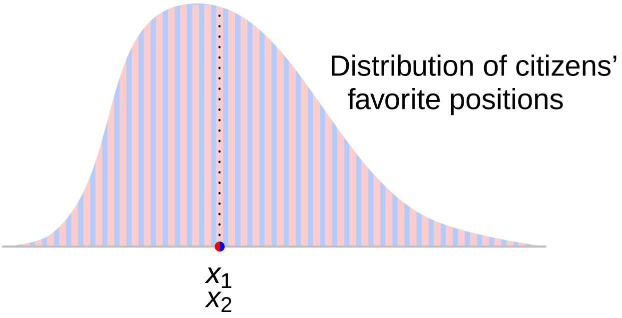
- Fix position x_1 of party 1
- If $x_1 \neq m$, best position for party 2: any position x_2 closer to m than x_1, where party 2 wins



Example: electoral competition

Subgame perfect equilibrium: subgames of length 1

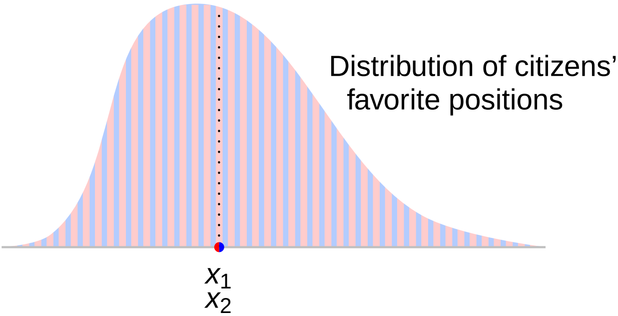
- Fix position x_1 of party 1
- If $x_1 \neq m$, best position for party 2: any position x_2 closer to m than x_1, where party 2 wins
- If $x_1 = m$, best position for party 2: m, where it ties



Example: electoral competition

Subgame perfect equilibrium: subgames of length 2

- Given party 2's reactions to party 1's position:
  - if party 1 chooses $x_1 \neq m$ then it loses
  - if party 1 chooses $x_1 = m$ then it ties
- Hence optimal position for party 1 at start of game: m



Example: electoral competition

Subgame perfect equilibrium: conclusion

- Game has many subgame perfect equilibria
- Every equilibrium has the form
  - $s_1(\varnothing) = m$: party 1 chooses m at start of game
  - $$s_2(x_1) \begin{cases} = m & \text{if } x_1 = m \\ \text{is closer to } m \text{ than is } x_1 & \text{if } x_1 \neq m \end{cases}$$
- One equilibrium: party 1 chooses m and party 2 chooses m after *every* history x_1 (i.e. for every position of party 1)
- Another equilibrium: party 1 chooses m and party 2 chooses midpoint of x_1 and m after history x_1
- In every equilibrium, *outcome* is that party 1 chooses m and then party 2 chooses m

Example: electoral competition

Strategic form of game

- Strategic form of extensive game is *not* the same as strategic game that models a situation in which parties choose simultaneously

Strategic game modeling simultaneous choice

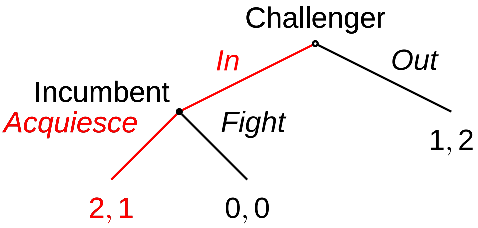
Players	Parties
Actions	For each party, set of possible positions

Strategic form of extensive game

Players	Parties
Actions	For party 1, set of possible positions; for party 2, *function* specifying position for each possible position of party 1

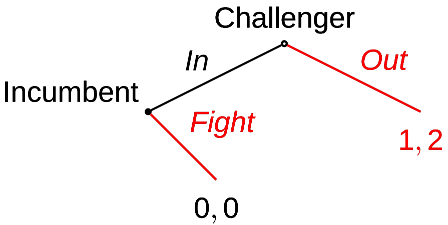
Are more options better?

- For an isolated decision-maker, more options are always better (ignoring the cost of making a decision!)
- Is the same true in a game?
- Consider entry game
- Unique subgame perfect equilibrium is $(In, Acquiesce)$



Are more options better?

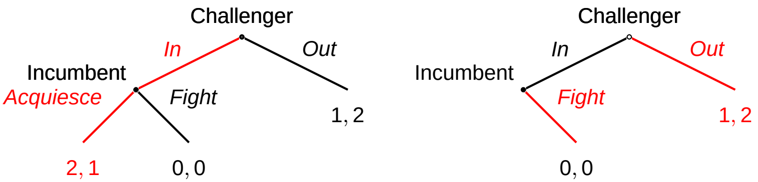
- Eliminate incumbent's option to acquiesce



- Then subgame perfect equilibrium: (*Out*, *Fight*)
- Incumbent is *better off* in this equilibrium than in equilibrium of original game
- So fewer options can be better
- Alternatively, commitment has a value
  - without option to acquiesce, threat to fight is credible

In surrounding an enemy, leave him a way out

- From Challenger's point of view, it is *better* for Incumbent to have option to acquiesce



- In this case, Incumbent's having more options is better for Challenger
- Corresponds to Sun Tzu's advice in *The Art of Warfare* (written between 500 BC and 300 BC): "in surrounding the enemy, leave him a way out; do not press an enemy that is cornered"