Economics 316

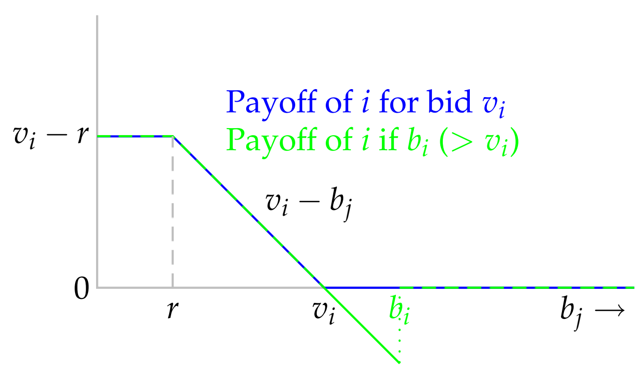
Fall 2017 Martin J. Osborne

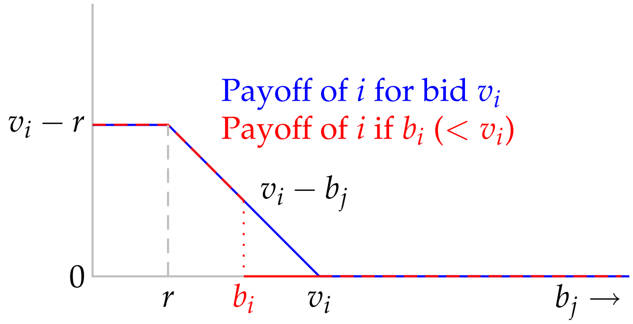
Solutions to Problem Set 8

1. In the first-price auction, a bidder with valuation v bids $\frac{1}{2}v$. A bidder wins if her bid is greater than the other player's bid. Given that $\frac{1}{2}v_i > \frac{1}{2}v_j$ if and only if $v_i > v_j$, a bidder with valuation v wins if the other player's valuation is less than v; the probability of that event is v (given the uniform distribution of valuations). Thus the expected price she pays is $\frac{1}{2}v^2$.

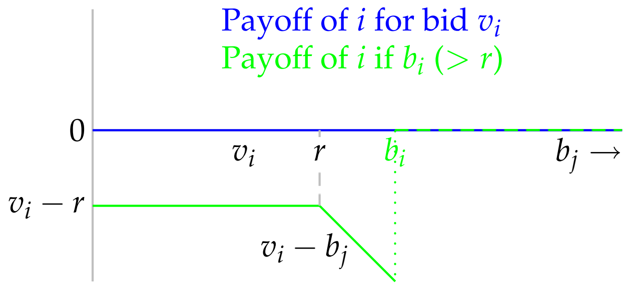
   In the second-price auction, a bidder with valuation v bids v and, if she wins, pays the second-highest price, the expected value of which is $\frac{1}{2}v$. Thus the expected price she pays is again $\frac{1}{2}v^2$.

2. (a) The payoffs of player i for the case $v_i > r$ are shown in the following two figures. In this case, any bid different from v_i is weakly dominated by a bid of v_i.





The payoffs for player i for the case $v_i < r$ and $b_i > r$ are shown in the following figure. In this case, a bid greater than r is weakly dominated by a bid of v_i (or indeed by any bid less than r). Any bid less than r is equivalent to a bid of v_i because in both cases player i does not win for any value of b_j, so that her payoff is zero for all values of b_j.



(b) Suppose that $v_i > r$. Then the expected price is

$$\begin{aligned}\Pr(v_j < r)r + \Pr(r < v_j < v_i)\tfrac{1}{2}(r + v_i) &= r^2 + \tfrac{1}{2}(v_i - r)(v_i + r) \\ &= \tfrac{1}{2}(v_i^2 + r^2).\end{aligned}$$

(c) We have

$$\begin{aligned}\int_r^1 \pi(v_i)\,dv_i &= \tfrac{1}{2}\int_r^1 (v_i^2 + r^2)\,dv_i \\ &= \tfrac{1}{2}[\tfrac{1}{3}v_i^3 + r^2 v_i]_r^1 \\ &= \tfrac{1}{2}(\tfrac{1}{3} + r^2 - \tfrac{4}{3}r^3).\end{aligned}$$

The maximizer of this function is the interior value of r for which its derivative is zero, which is $r = \frac{1}{2}$. (Sketch the function.)

Thus the optimal reserve price is $\frac{1}{2}$.

3. Suppose that player 2 with valuation v_2 bids βv_2^2. Consider player 1 with valuation v_1. If she bids b_1, then she wins if $\beta v_2^2 < b_1$, or if $v_2 < \sqrt{b_1/\beta}$, which has probability $\sqrt{b_1/\beta}$ if $b_1 \leq \beta$ and probability 1 if $b_1 > \beta$. Thus her payoff is

$$v_1\sqrt{b_1/\beta} - b_1 \text{ if } b_1 \leq \beta$$

and $v_1 - b_1$ if $b_1 > \beta$ (remember that she pays b_1 whether or not she wins). This payoff is maximized when

$$b_1 = v_1^2/(4\beta).$$

For a symmetric equilibrium, the players' strategies have to be the same, so that we need $\beta = 1/(4\beta)$, or $\beta = \frac{1}{2}$.

Thus the auction has an equilibrium in which each player i with valuation v_i bids $\frac{1}{2}v_i^2$.

The expected price paid by a bidder with valuation v is simply her bid, because she pays this amount whether or not she wins. Thus the expected price paid is $\frac{1}{2}v^2$, as it is in the first- and second-price auctions.

4. (a) If player 2 wins, she knows that player 1 has bid at most 1, implying that the painting is fake.

I now argue that the strategy pair is not an equilibrium because player 2 can profitably deviate. Given the strategy pair, if the painting is fake then player 2 wins (she bids $x_2 + 5$, which is at least 5, and player 1 bids x_1, which is at most 1) and pays x_1, and if the painting is authentic then player 2 loses (she bids $x_2 + 5$, which is at most 6, and player 1 bids $x_1 + 10$, which is at least 10). Thus player 2's expected payoff to the strategy pair is the expected value of $\frac{1}{2}(x_2 - x_1)$. The expected value of the sum of two random variables is the sum of their expected values (that's a general result), so the expected value of $x_1 - x_2$ is the difference between the expected value of x_1, which is $\frac{1}{2}$, and the expected value of x_2, which is also $\frac{1}{2}$. So the expected value of $\frac{1}{2}(x_2 - x_1)$ is 0.

If instead player 2 bids x_2 then, given player 1's strategy, she wins only if the painting is fake and $x_1 < x_2$, in which case her payoff is positive (she pays x_1 and her valuation is x_2). So by deviating to this strategy, player 2 is better off (she avoids winning when the painting is fake and $x_2 < x_1$, in which case she pays more than her valuation).

(b) The auction has an equilibrium in which player 1 bids her valuation, which is x_1 if the painting is fake and $10 + x_1$ if the painting is authentic, and player 2 of type x_2 bids x_2.

The outcome of this strategy pair is that if the painting is authentic, player 1 wins and pays x_2 (a bargain!), and if the painting is fake, player 1 wins and pays x_2 if $x_1 > x_2$ and player 2 wins and pays x_1 if $x_2 > x_1$.

For player 1, bidding her valuation is her only weakly undominated actions, as in the second-price auction we considered in class.

For player 2 of type x_2, raising her bid to any $b_2 \leq 10$ reduces her expected payoff, because she then wins in some cases in which x_1, which is the price she pays, exceeds her valuation x_2, and never wins when the painting is authentic. If she raises her bid above 10 then the best bid is $x_2 + 10$, her valuation if the painting is authentic. Her expected payoff if the painting is fake is then zero and her expected payoff if the painting is authentic is the same as her expected payoff if it is fake and she bids x_2. Thus she cannot increase her expected payoff by raising her bid. If she reduces her bid then she reduces her payoff because she then fails to win in some cases in which player 1's bid (and hence the price if she were to win) is less than x_2.

The game has another equilibrium in which player 1 bids her valuation, which is x_1 if the painting is fake and $10 + x_1$ if the painting is authentic, and player 2 of type x_2 bids $x_2 + 10$. (The argument is similar to the previous argument.) Is this strategy of player 2 weakly dominated?

5. First consider a first-price auction. The players' payoffs when player 2 has valuation 0 are given in the table in left-hand panel of Figure 1. Each row is a bid of player 1 and each column is a bid of player 2. We see that player 2's bid of 1 is strictly dominated by her bid of 0 in this case. Thus in any equilibrium of the whole game, she bids 0.

	0	1
0	$\frac{1}{2}, 0$	$0, -1$
1	$0, 0$	$0, -\frac{1}{2}$

Player 2's valuation is 0

	0	1
0	$\frac{1}{2}, \frac{1}{2}$	$0, 0$
1	$0, 0$	$0, 0$

Player 2's valuation is 1

Figure 1. The players' payoffs in a first-price auction.

The players' payoffs when player 2 has valuation 1 are given in the table in the right-hand panel of the figure. We see that player 1's bid of 0 weakly dominates her bid of 1 (taking into account both possible valuations for player 2). (If she bids 1 then her payoff is 0 regardless of player 2's valuation and bid.)

We conclude that in any equilibrium in which neither player uses a weakly dominated strategy, player 1 bids 0 and player 2 bids 0 when her valuation is 0. Consequently, player 2 bids 0 when her valuation is 1 (doing so is her only best response to player 1's bid of 0). The auctioneer's revenue in this equilibrium is 0.

Now consider a second-price auction. As in any second-price auction, the only strategy of each player that is not weakly dominated is the strategy of bidding her valuation. (If you want to check that specifically for this auction, look at the players' payoffs, which are given in Figure 2.) The resulting strategy pair is a Nash equilibrium. In this equilibrium the auctioneer's revenue is $1 - p$ (the probability that player 2's valuation is 1, in which case the price is 1).

	0	1
0	$\frac{1}{2}, 0$	$0, 0$
1	$1, 0$	$0, -\frac{1}{2}$

Player 2's valuation is 0

	0	1
0	$\frac{1}{2}, \frac{1}{2}$	$0, 1$
1	$1, 0$	$0, 0$

Player 2's valuation is 1

Figure 2. The players' payoffs in a second-price auction.

We conclude that the auctioneer's revenue is higher in the second-price auction than it is in the first-price auction.