Economics 316

Fall 2017 | Martin J. Osborne

Problem Set 8

1. Find the expected price paid by a bidder with valuation v in the equilibrium of the example of a first-price auction studied in class, with two bidders each of whose valuations are uniformly distributed on $[0, 1]$. Verify that this expected price is the same as it is in the equilibrium of the second-price auction, as the revenue equivalence theorem implies.

2. In many auctions, the auctioneer sets a "reserve price"—a price below which she will not sell the object. Consider a second-price sealed-bid auction with two bidders, each of whose valuations is distributed uniformly on $[0, 1]$ (as in the example in class). Denote by r the reserve price. If both bids are greater than r, then the player who submits the higher bid wins and pays a price equal to the lower bid. If one bid is greater than r and the other is less than r, then the player who submits the bid greater than r wins and pays r. If both bids are less than r, then the object is not sold.

   Find the value of r that maximizes the expected revenue of the auctioneer, as follows.

   (a) Show that (i) for a player whose valuation exceeds r, the strategy of bidding her valuation weakly dominates every other strategy, and (ii) for a player whose valuation is less than r, the strategy of bidding her valuation weakly dominates any strategy of bidding more than r and is equivalent to every other strategy.

   (b) Find the expected price paid by player i with valuation v_i who bids v_i. If $v_i < r$, then of course this expected price is 0 (the bidder never obtains the object). Now suppose that $v_i \geq r$. Then with probability r, the other player's valuation is less than r, in which case the player pays r, and with probability $1 - v_i$ the other player's valuation is greater than v_i, in which case the player does not win. In the remaining case, the other player's valuation is between r and v_i. This case occurs with probability $v_i - r$, and the other player's valuation is uniformly distributed between r and v_i. Thus the expected price paid by the player in this case is $\frac{1}{2}(r + v_i)$. If you put all these cases together, you get the expected price paid by a player with valuation v_i.

(c) Denote the expected price paid by player i with valuation v_i by $\pi(v_i)$. The expected revenue of the auctioneer is twice the expected value of $\pi(v_i)$. (Twice because there are two bidders.) The valuation v_i is uniformly distributed on $[0, 1]$, and $\pi(v_i) = 0$ if $v_i < r$, so the expected value of $\pi(v_i)$ is

$$\int_r^1 \pi(v_i)\, dv_i.$$

Calculate this expected value, and then find the value of r that maximizes the auctioneer's revenue.

3. Consider a variant of the example of a first-price auction studied in class in which *all* bidders, not only the winner, pay their bids. (Such an "all-pay" auction models lobbying, for example. Every lobbyist exerts effort, and the one who exerts the most effort wins.) Suppose there are two bidders, with valuations distributed uniformly on $[0, 1]$. Suppose that player 2 with valuation v_2 bids βv_2^2. (Notice the square.) Show that the optimal bid of player 1 with valuation v_1 is $\frac{1}{4}v_1^2/\beta$, and thus find an equilibrium of the auction.

   Calculate the expected price paid by a bidder with valuation v. Compare this expected price with the expected price paid by a bidder with valuation v in a first- or second-price auction.

4. Consider an example of a second-price auction with common values in which a painting is for sale and that painting may be a fake. There are two bidders; bidder 1 is an expert and bidder 2 is not. The expert knows whether the painting is fake, but bidder 2 does not. Bidder 2 believes that the probability the painting is fake is $\frac{1}{2}$. Both bidders' valuations contain a random component, as in the case of independent private valuations, but depend also on whether the painting is fake. Specifically, before the auction two numbers, x_1 and x_2, are drawn independently from a uniform distribution on $[0, 1]$. Bidder 1's valuation is x_1 if the painting is fake and $x_1 + 10$ if the painting is authentic, and bidder 2's valuation is x_2 if the painting is fake and $x_2 + 10$ if the painting is authentic.

(a) Suppose that each player bids her expected valuation for the painting. That is, player 1 bids x_1 if the painting is fake and $x_1 + 10$ if it is authentic, and player 2 bids $\frac{1}{2}(x_2 + (x_2 + 10)) = x_2 + 5$. If the players use this strategy pair and player 2 wins, what does she learn? Show that the strategy pair is not an equilibrium of the auction.

(b) When deciding how much to bid, player 2 (like all bidders in any auction) should consider her expected value of the object *conditional on winning*. That is, for each possible bid, she should ask herself what her expected valuation of the object would be if she were to win with that bid. Find an equilibrium of the auction.

5. Here is a simple example of an auction in which not all bidders' valuations are drawn from the same distribution, to illustrate how in such a case first- and second-price auctions may not be revenue equivalent. Player 1's valuation of the object for sale is 1 and is known by player 2. Player 2's valuation is known by her but not by player 1, who believes it to be 0 with probability p and 1 with probability $1 - p$, where $0 < p < 1$. The player who bids the most obtains the object and pays either the price she bids ("first-price auction") or the highest of the remaining bids ("second-price auction"). If there is a tie for the highest bid, each player obtains the object with probability $\frac{1}{2}$, and only the player who does so makes a payment. Each player's payoff if she obtains the object is the expected value of the difference between her valuation and the amount she pays; her payoff if she does not obtain the object is 0.

   Study the equilibria of each game *when each player is restricted to bid either 0 or 1* and uses a strategy that is not weakly dominated. Compare the auctioneer's revenue from the two auctions.