ECO316: Applied game theory

Lecture 8

Martin J. Osborne

Department of Economics
University of Toronto

2017.10.26

Table of contents

Auctions

Many types of goods are transacted by auction

- Art
- Fish, cattle, flowers
- Treasury bills
- Oil tracts, timber
- Wireless spectrum (for cell phones, TV, . . .): revenue from 2008 auction in Canada was \$4.25 billion
- Ads on search engines (Google ad revenue second quarter of 2016 about \$19 billion)
- eBay (2015 sales revenue \$22 billion)
- Government contracts
- Repairs to your house

Auctions

- Traders submit bids
- Winner and price depend on bids
- Many types of auction
  - Ascending auction (English auction): bids sequential, price rises, stops when no one wants to bid higher than current bid (art, fish, . . .)
  - Descending auction (Dutch auction): bids sequential, price falls, first bidder wins (flowers in Netherlands)
  - Bids simultaneous (sealed-bid): winner is highest bidder, price may be highest bid or other price (eBay, government contracts, ads on Google, oil tracts)
  - We will study model of sealed-bid auction
- Single item for sale, or multiple interconnected items
  - Will consider case of *single* item for sale

Auctions

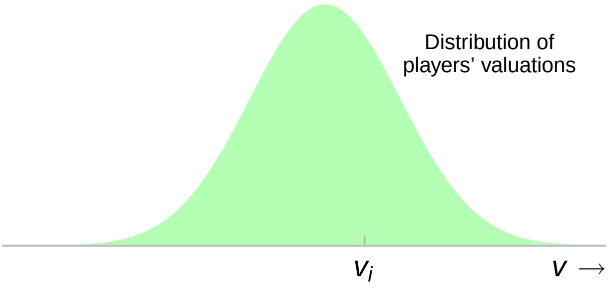
Independent private valuations

- Each bidder knows her own valuation of the object for sale
- . . . but does not know the *other* bidders' valuations
- Each bidder knows the *distribution* of the other bidders' valuations
- Specifically, each bidder knows that each other bidder's valuation is drawn independently from same distribution, independently of her own valuation
- Known as independent private valuations

Independent private valuations sealed-bid auction

Strategic game with imperfect information

- Players: the bidders
- Each player i
  - has many possible types—one for each possible valuation
  - knows her own valuation but not other players' valuations
  - believes that other bidders' valuations are drawn independently from same distribution, independently of her own valuation



Independent private valuations sealed-bid auction

Strategic game with imperfect information

- Players: the bidders
- Each player i
  - has many possible types—one for each possible valuation
  - knows her own valuation but not other players' valuations
  - believes that other bidders' valuations are drawn independently from same distribution, independently of her own valuation
- For each player i,
  - actions are possible bids (numbers)
  - payoff is
    - $v_i - p(b_1, \ldots, b_n)$ if i's bid b_i is higher than bid b_j of every other player j, where $p(b_1, \ldots, b_n)$ is price paid
    - 0 otherwise

  where v_i is i's valuation

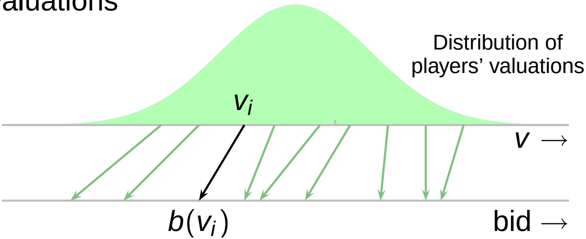
Independent private valuations

Strategies

A player's strategy specifies a bid for *each* of her possible valuations (types)

Nash equilibrium

A strategy profile is a Nash equilibrium if the bid specified by each player's strategy for each of her possible valuations (types) maximizes the player's expected payoff *given* the other players' strategies and the player's belief about the other players' valuations



Independent private valuations

- The strategic game explicitly models a sealed-bid auction
- But we now argue that it can be used also to model ascending and descending auctions

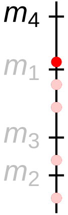
Ascending auctions

- What instructions would you give to someone to bid on your behalf?
- Give them a maximum amount you are willing to pay, independent of other bids?
- Makes sense given that your payoff is independent of everyone else's valuations

Independent private values

Ascending auction

- If every bidder's strategy is to bid up to some maximum amount, what is outcome?
- Suppose 4 bidders with limit bids m_1, m_2, m_3, and m_4
- Price starts low: everyone wants to bid
- As price rises, bidders drop out
- Once price goes above m_1, bidding stops $\Rightarrow$ bidder 4 wins and pays price slightly above m_1—*second highest* limit bid



Independent private values

So we can model an ascending auction as the strategic game with imperfect information we have defined in which

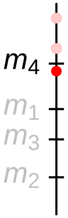
- each player's action is a limit bid (highest she is willing to pay)
- winner is player with highest limit bid
- price is *second* highest limit bid

This model is second-price sealed-bid auction

Independent private values

Descending (Dutch) auction

- In descending auction, price starts high and falls until someone bids
- Suppose 4 bidders with limit bids m_1, m_2, m_3, and m_4
- Price starts high: no one wants to bid
- When price falls below highest limit bid, bidding stops $\Rightarrow$ bidder 4 wins and pays price slightly below her limit bid, m_4



Independent private values

So we can model a descending auction as the strategic game with imperfect information we have defined in which

- each player's action is a limit bid (highest she is willing to pay)
- winner is player with highest limit bid
- price is *highest* limit bid

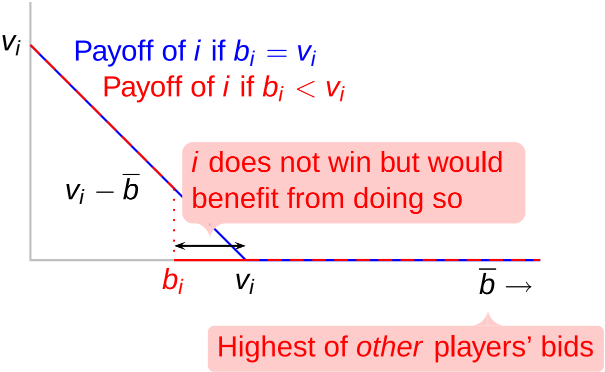
This model is first-price sealed-bid auction

Second-price sealed-bid auction

- Single object for sale
- n bidders
- Each bidder's valuation of object known to her, fixed independently of other bidders' valuations
- Each bidder doesn't know *other* bidders' valuations; believes each is drawn independently from same distribution
- Bids submitted simultaneously
- Bidder who submits highest bid wins
- Price paid by winner is highest losing bid (if there are no ties, second highest bid)

Second-price sealed-bid auction

Analysis

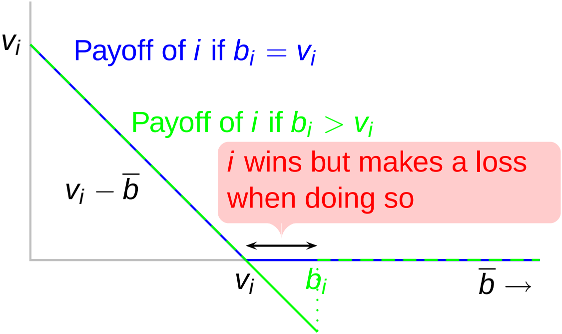


$\Rightarrow$ bid of v_i weakly dominates bid of $b_i < v_i$

Note: bidding less in a second-price auction does not affect the price—only possibly the probability of winning

Second-price sealed-bid auction

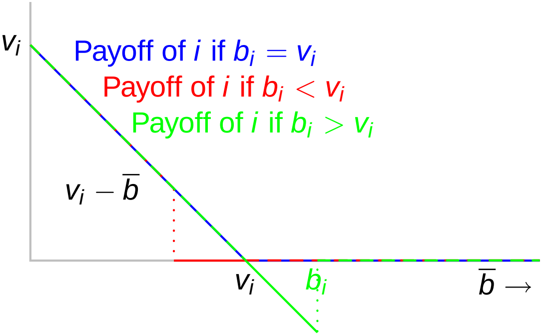
Analysis



$\Rightarrow$ bid of v_i weakly dominates bid of $b_i > v_i$

Second-price sealed-bid auction

Analysis



Proposition

For player i with valuation v_i in a second-price sealed-bid auction, the bid v_i weakly dominates all other bids.

Second-price sealed-bid auction

Because a player's bidding her valuation weakly dominates all her other actions

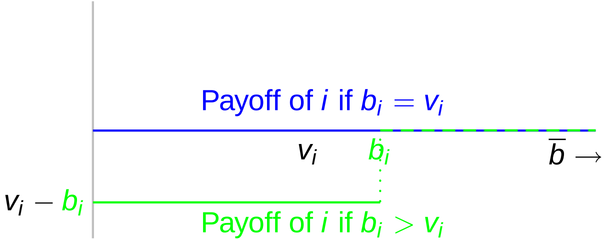
Proposition

An independent private values second-price sealed-bid auction has a Nash equilibrium in which every player bids her valuation

- Object is sold to bidder with highest valuation
- Auction has *other* equilibria too, but we focus on this one
- Remember that second-price auction models open ascending auction and eBay auction

First-price sealed-bid auction

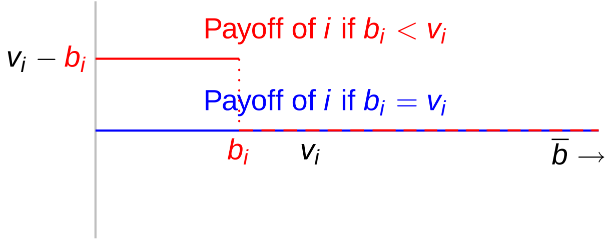
Analysis



$\Rightarrow$ bid of v_i weakly dominates bid of $b_i > v_i$

First-price sealed-bid auction

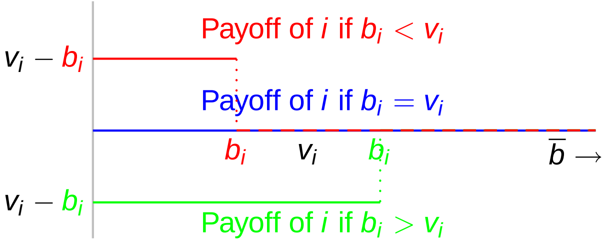
Analysis



$\Rightarrow$ bid of v_i *is weakly dominated by* bid of $b_i < v_i$

First-price sealed-bid auction

Analysis



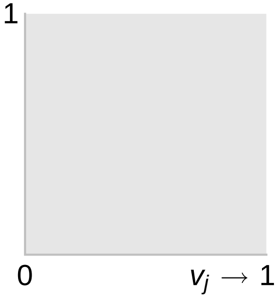
Proposition

For player i with valuation v_i in a first-price sealed-bid auction, the bid v_i weakly dominates all higher bids but does **not** weakly dominate lower bids. In fact, the bid v_i is weakly dominated by any lower bid.

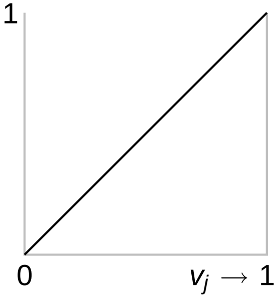
First-price sealed-bid auction

Example

- Two bidders
- Each bidder i believes that other bidder's valuation is distributed uniformly between 0 and 1, independently of v_i



Probability density of v_j



Cumulative distribution of v_j

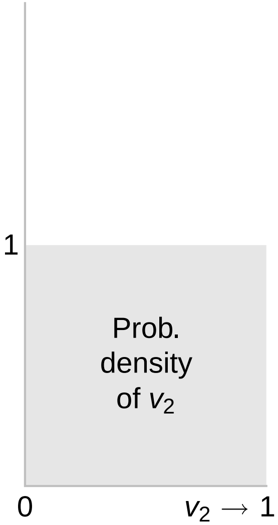
First-price sealed-bid auction

Analysis of example

Claim The auction has a (symmetric) Nash equilibrium in which a player with valuation v bids $\frac{1}{2}v$. That is, each player bids exactly half her valuation.

Argument

- Suppose that for each valuation v_2, bid of player 2 with valuation v_2 is $\frac{1}{2}v_2$
- Then player 1 believes that player 2's bids are distributed uniformly between 0 and $\frac{1}{2}$



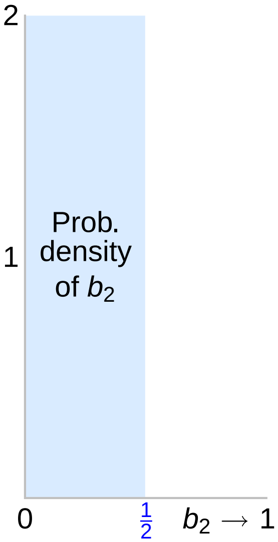
First-price sealed-bid auction

Analysis of example

Claim The auction has a (symmetric) Nash equilibrium in which a player with valuation v bids $\frac{1}{2}v$. That is, each player bids exactly half her valuation.

Argument

- Suppose that for each valuation v_2, bid of player 2 with valuation v_2 is $\frac{1}{2}v_2$
- Then player 1 believes that player 2's bids are distributed uniformly between 0 and $\frac{1}{2}$

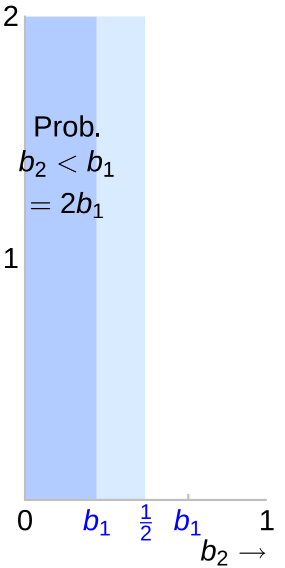


First-price sealed-bid auction

Analysis of example

- Suppose player 1 bids b_1
- $b_1 > \frac{1}{2} \Rightarrow$ player 1 surely wins
- $b_1 \leq \frac{1}{2} \Rightarrow$ prob. player 1 wins $= \Pr(b_2 < b_1) = 2b_1$
- So player 1's payoff is

$$\begin{cases} 2b_1(v_1 - b_1) & \text{if } 0 \leq b_1 \leq \frac{1}{2} \\ v_1 - b_1 & \text{if } b_1 > \frac{1}{2} \end{cases}$$

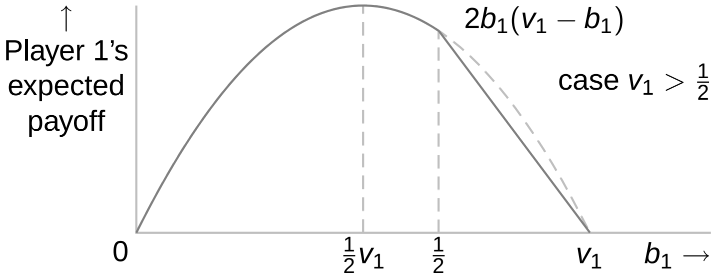


First-price sealed-bid auction

Analysis of example

- So far: if, for each valuation v_2, player 2's bid is $\frac{1}{2}v_2$, then expected payoff of player 1 with valuation v_1 to bidding b_1 is

$$\begin{cases} 2b_1(v_1 - b_1) & \text{if } 0 \leq b_1 \leq \frac{1}{2} \\ v_1 - b_1 & \text{if } b_1 > \frac{1}{2} \end{cases}$$



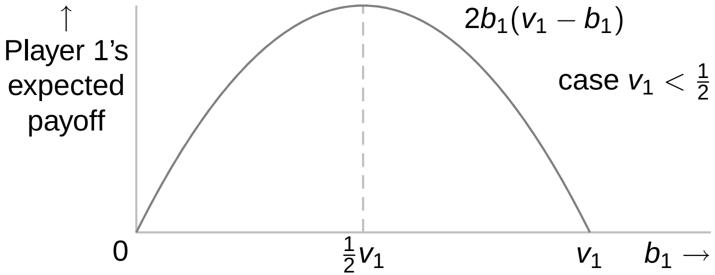
- Thus best response of player 1 with valuation v_1 is $\frac{1}{2}v_1$

First-price sealed-bid auction

Analysis of example

- So far: if, for each valuation v_2, player 2's bid is $\frac{1}{2}v_2$, then expected payoff of player 1 with valuation v_1 to bidding b_1 is

$$\begin{cases} 2b_1(v_1 - b_1) & \text{if } 0 \leq b_1 \leq \frac{1}{2} \\ v_1 - b_1 & \text{if } b_1 > \frac{1}{2} \end{cases}$$



- Thus best response of player 1 with valuation v_1 is $\frac{1}{2}v_1$

First-price sealed-bid auction

Example: conclusion

- If, for each valuation v_2, player 2 with valuation v_2 bids $\frac{1}{2}v_2$, then best bid for player 1 with valuation v_1 is $\frac{1}{2}v_1$
- Argument for player 2 is symmetric: if, for each valuation v_1, player 1 with valuation v_1 bids $\frac{1}{2}v_1$ then best bid of player 2 with valuation v_2 is $\frac{1}{2}v_2$
- So auction has Nash equilibrium in which each player with valuation v_i bids $\frac{1}{2}v_i$
- In this equilibrium, as in equilibrium in second-price auction, object is sold to bidder with highest valuation

First-price sealed-bid auction

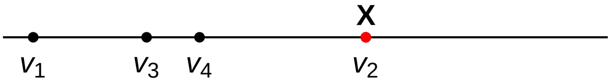
Example: interpretation of equilibrium bids

- When player i has valuation v_i, she wins if and only if player j's valuation v_j is less than v_i: $0 \leq v_j \leq v_i$
- Over all cases in which player j's valuation is less than v_i, expected value of player j's valuation is $\frac{1}{2}v_i$
- Thus equilibrium bid of player i with valuation v_i is expected value of player j's valuation conditional on that valuation being less than v_i: $\mathrm{E}(v_j \mid v_j < v_i)$

First-price sealed-bid auction

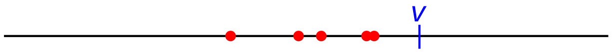
Generalizing the example

- Suppose there are n bidders
- Choose $n-1$ valuations randomly and independently (remember that every bidder believes every other bidder's valuation is drawn independently from same distribution)
- The highest of these $n-1$ valuations is a random variable: its value depends on the $n-1$ valuations that were chosen
- Denote the highest of the $n-1$ valuations by **X**



First-price sealed-bid auction

Generalizing the example



- Fix a valuation v
- Some values of $\mathbf{X}$ are less than v; others are greater than v
- Consider the distribution of $\mathbf{X}$ in those cases in which $\mathbf{X} < v$
- The expected value of this distribution is denoted $E(\mathbf{X} \mid \mathbf{X} < v)$: the expected value of $\mathbf{X}$ conditional on $\mathbf{X}$ being less than v

First-price sealed-bid auction

Analogy: birthdays

- Birthdays (month, day) in group of people are more or less uniformly distributed from January 1 to December 31
- Expected birthday is around July 1
- Let **X** be latest birthday in group
- What is $\mathrm{E}(\mathbf{X} \mid \mathbf{X} < \text{May } 1)$: expected *latest* birthday before May 1?
- If one person in group, $\mathrm{E}(\mathbf{X} \mid \mathbf{X} < \text{May } 1)$ is around March 1
- As size of group increases, $\mathrm{E}(\mathbf{X} \mid \mathbf{X} < \text{May } 1)$ approaches April 30

First-price sealed-bid auction

Generalizing the example

Proposition

For any number of bidders and any distribution of valuations, a first-price sealed-bid auction has a Nash equilibrium in which a player with valuation v bids $\mathrm{E}(\mathbf{X} \mid \mathbf{X} < v)$, the expected value of the highest of the other players' valuations conditional on v being higher than all the other valuations.

- That is, when deciding how much to bid, each bidder asks: Over all cases in which my valuation is the highest, what is expectation of highest of other players' valuations?
- This expectation is the amount she bids
- Comparative static: more bidders $\Rightarrow \mathrm{E}(\mathbf{X} \mid \mathbf{X} < v)$ higher $\Rightarrow$ bids higher
- Very large number of bidders $\Rightarrow \mathrm{E}(\mathbf{X} \mid \mathbf{X} < v)$ close to v

Comparison of first- and second-price auctions

First-price auction

- Bidder with valuation v bids $\mathrm{E}(\mathbf{X} \mid \mathbf{X} < v)$
- Winner is bidder with highest valuation v^*, who pays $\mathrm{E}(\mathbf{X} \mid \mathbf{X} < v^*)$

Second-price auction

- Bidder with valuation v bids v
- Winner is bidder with highest valuation v^*, who pays price equal to second-highest bid, the expected value of which is $\mathrm{E}(\mathbf{X} \mid \mathbf{X} < v^*)$

Proposition (*Revenue equivalence*)

Under our assumptions, first- and second-price auctions yield the same expected revenue

Note: assumptions include risk-neutrality (payoffs $v - p$)

Auctions with private valuations: Summary

Second-price auction

- Bid $=$ valuation weakly dominates every other bid
- Equilibrium in which every player bids her valuation

First-price auction

- Bid $=$ valuation weakly dominates higher bids, but not lower bids
- Equilibrium in which every player bids less than her valuation
- The more bidders there are, the higher the bids

Revenue equivalence

- Expected revenue of seller is same in first- and second-price auctions

Common value auctions

- In many auctions, bidders' valuations are not independent
- Instead, bidders' valuations may be related to each other
- Even a buyer of a work of art may care about its resale value, which depends on other people's valuations of it
- Interdependence of values introduces considerations not present when values are independent

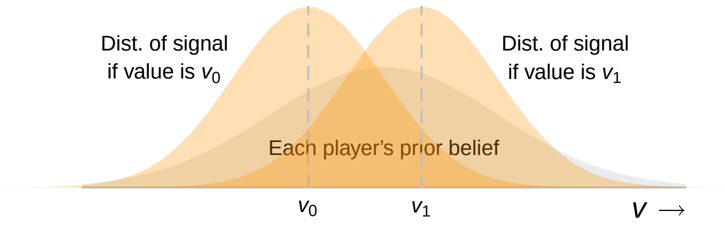
Common value auctions

Drilling for oil

- All firms value oil in the same way
- But no firm knows amount available
- Each firm privately takes a sample, which provides information about amount available
- Samples differ, so estimates of amount available based on samples differ

Common value auctions

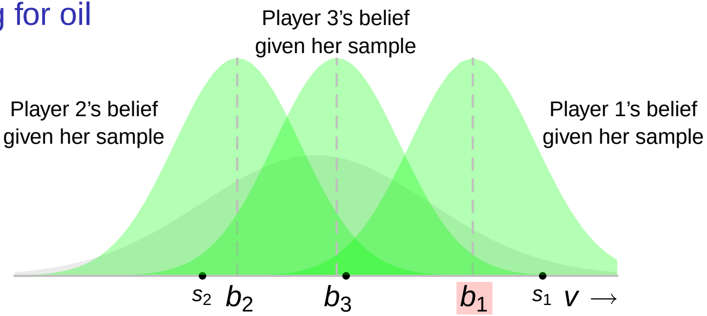
Drilling for oil



- Each player sees only her *own* sample
- On basis of sample and prior belief, forms probabilistic estimate of value
- Different players get different samples and form different estimates
- Suppose each player bids according to her own estimate

Common value auctions

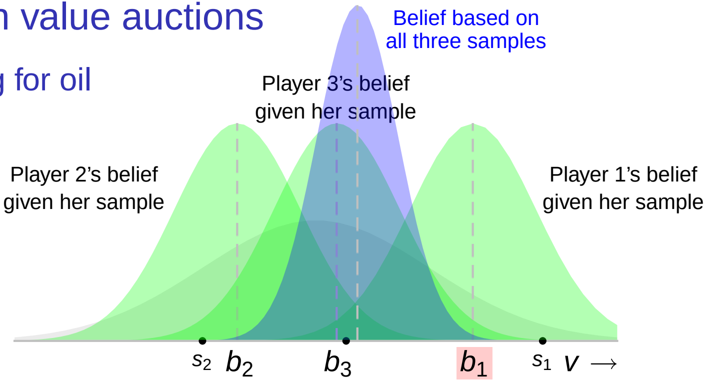
Drilling for oil



- Each player sees only her *own* sample
- On basis of sample and prior belief, forms probabilistic estimate of value
- Different players get different samples and form different estimates
- Suppose each player bids according to her own estimate

Common value auctions

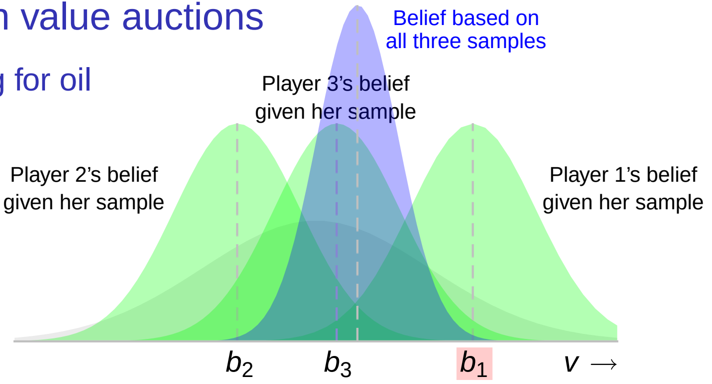
Drilling for oil



- Highest bid wins $\Rightarrow$ player with best sample wins
- *Best* sample typically overestimates value $\Rightarrow$ winner incurs loss
- Effect is known as winner's curse

Common value auctions

Drilling for oil



- If player wins, she knows other players' estimates of value are lower than hers
- She should taken this information into account, and base her bid on estimate of value *conditional on winning* (given other players' strategies)

Common value auctions

Getting your roof fixed

- You get three estimates to fix your roof
- First roofer: thinks the bit around the chimney will be hard to do; gives you estimate of $3,000
- Second roofer: correctly estimates the difficulty of the job; gives you estimate of $2,000
- Third roofer: doesn't notice the tricky bit at the back; gives you estimate of $1,500
- You choose the lowest estimate—the third one, based on an underestimate of the difficulty of the job
- The winner makes a loss
- The lesson: estimates should take into account that you will choose the *lowest* one

Common value auctions

Summary

- Even for second-price rule, not an equilibrium in common value auction for each player to bid her valuation based on her own information
- Players who do so suffer “winner’s curse”
- Instead, player should base her bid on her estimate of her valuation of the object over all cases in which all other bids are lower than hers, given the other players’ equilibrium strategies