ECO316: Applied game theory

Lecture 8

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Auctions

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- Art

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  - Will consider case of *single* item for sale

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Independent private valuations

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- Known as independent private valuations

Independent private valuations sealed-bid auction

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Each player can condition her bid on her valuation (which she knows)

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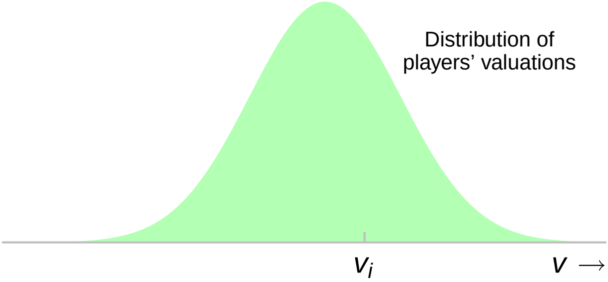

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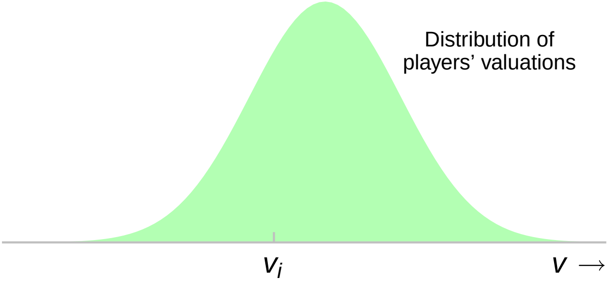

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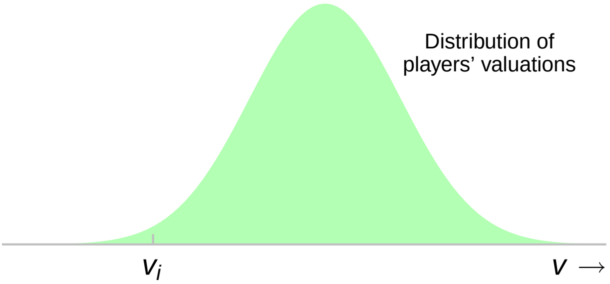

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    - $v_i - p(b_1, \ldots, b_n)$ if *i*'s bid b_i is higher than bid b_j of every other player *j*, where $p(b_1, \ldots, b_n)$ is price paid

Nature of *p* depends on auction rule

where v_i is *i*'s valuation

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    - 0 otherwise

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A player's strategy specifies a bid for *each* of her possible valuations (types)

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Nash equilibrium

A strategy profile is a Nash equilibrium if the bid specified by each player's strategy for each of her possible valuations (types)

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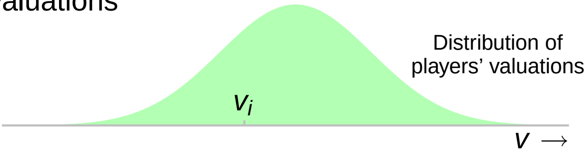

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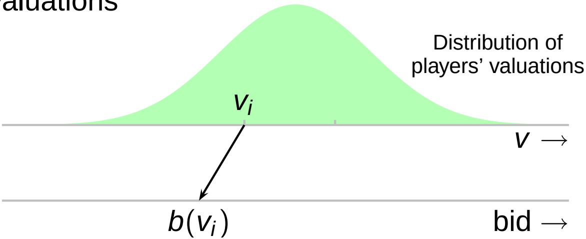

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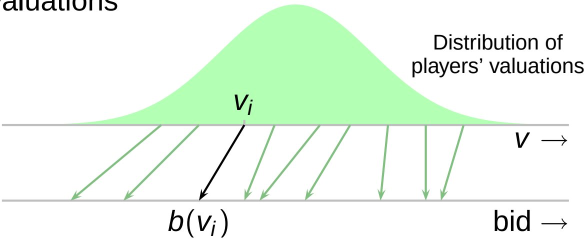

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- What instructions would you give to someone to bid on your behalf?
- Give them a maximum amount you are willing to pay, independent of other bids?
- Makes sense given that your payoff is independent of everyone else's valuations

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Ascending auction

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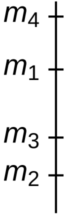

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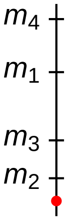

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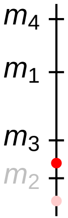

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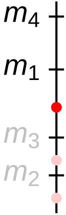

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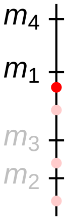

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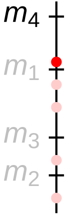

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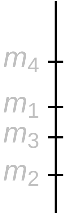

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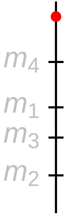

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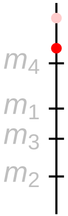

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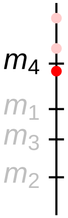

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Second-price sealed-bid auction

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- Price paid by winner is highest losing bid (if there are no ties, second highest bid)

Second-price sealed-bid auction

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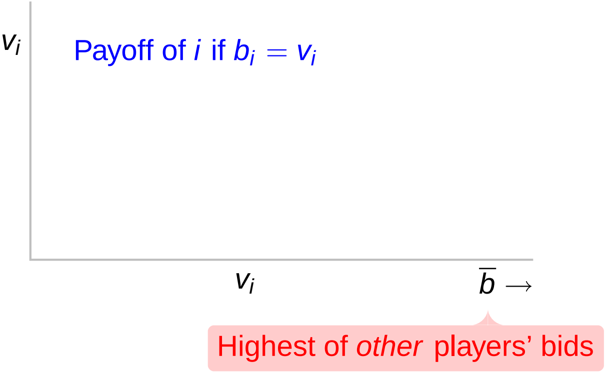

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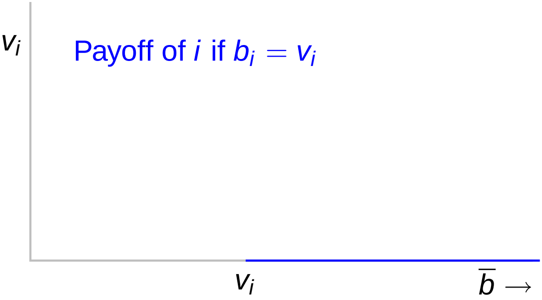

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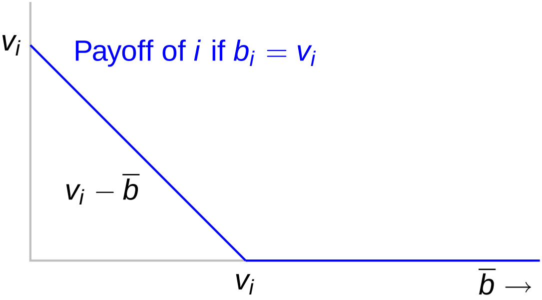

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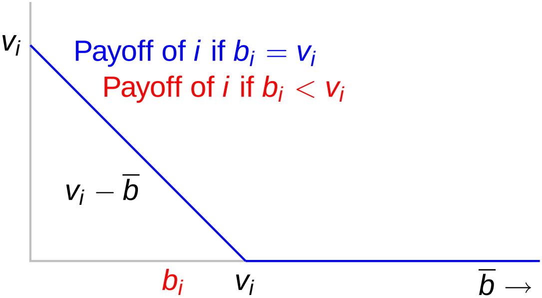

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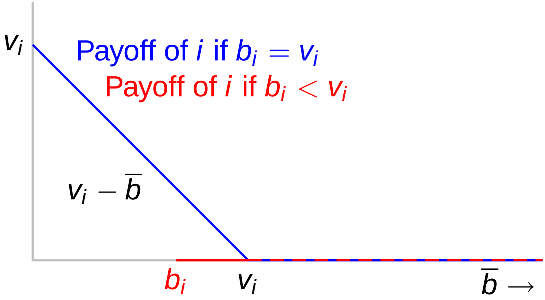
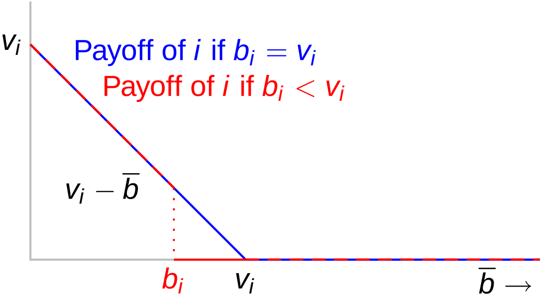


Second-price sealed-bid auction

Analysis



Second-price sealed-bid auction

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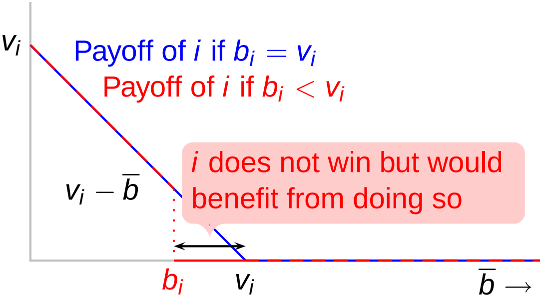
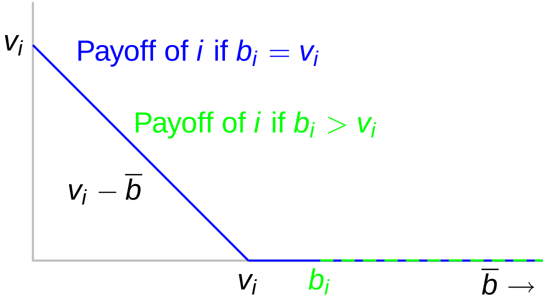


$\Rightarrow$ bid of v_i weakly dominates bid of $b_i < v_i$

Note: bidding less in a second-price auction does not affect the price—only possibly the probability of winning

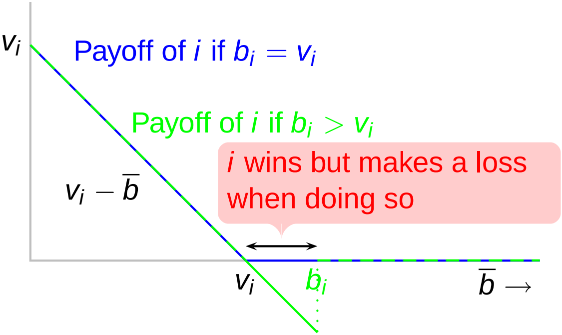
Second-price sealed-bid auction

Analysis



Second-price sealed-bid auction

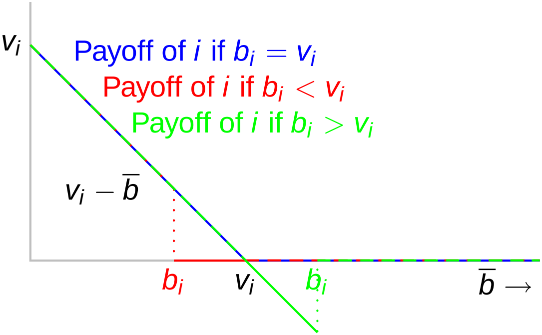
Analysis



$\Rightarrow$ bid of v_i weakly dominates bid of $b_i > v_i$

Second-price sealed-bid auction

Analysis



Proposition

For player i with valuation v_i in a second-price sealed-bid auction, the bid v_i weakly dominates all other bids.

Second-price sealed-bid auction

Because a player's bidding her valuation weakly dominates all her other actions

Proposition

An independent private values second-price sealed-bid auction has a Nash equilibrium in which every player bids her valuation

Second-price sealed-bid auction

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- Object is sold to bidder with highest valuation

Second-price sealed-bid auction

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- Object is sold to bidder with highest valuation
- Auction has *other* equilibria too, but we focus on this one

Second-price sealed-bid auction

Because a player's bidding her valuation weakly dominates all her other actions

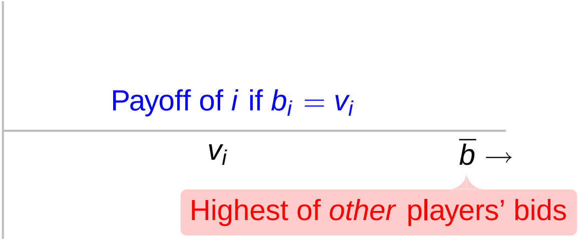
Proposition

An independent private values second-price sealed-bid auction has a Nash equilibrium in which every player bids her valuation

- Object is sold to bidder with highest valuation
- Auction has *other* equilibria too, but we focus on this one
- Remember that second-price auction models open ascending auction and eBay auction

First-price sealed-bid auction

Analysis



First-price sealed-bid auction

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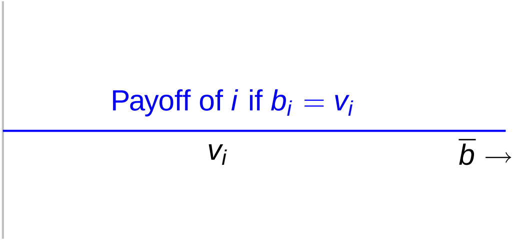

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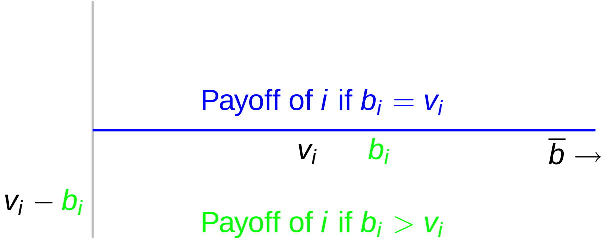

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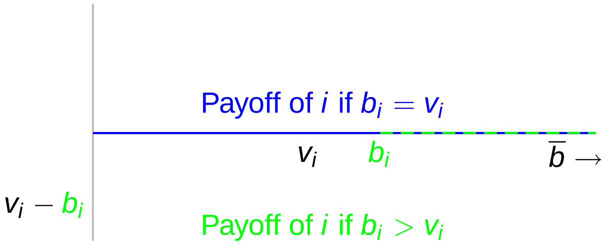

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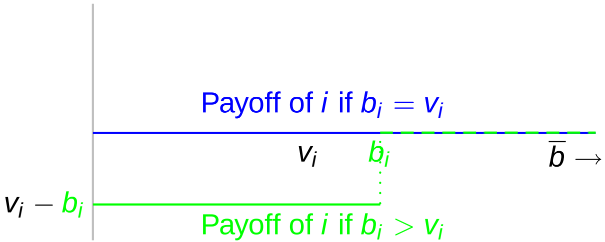

First-price sealed-bid auction

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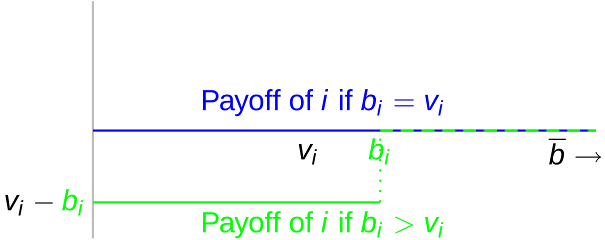
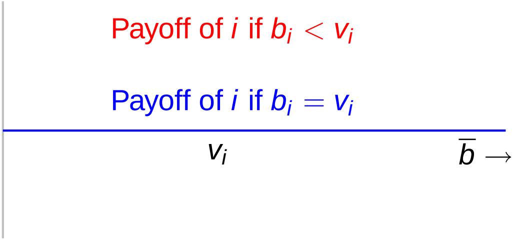


$\Rightarrow$ bid of v_i weakly dominates bid of $b_i > v_i$

First-price sealed-bid auction

Analysis



First-price sealed-bid auction

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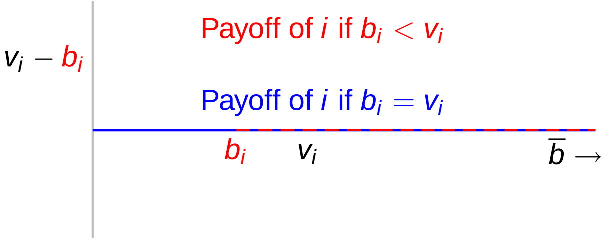

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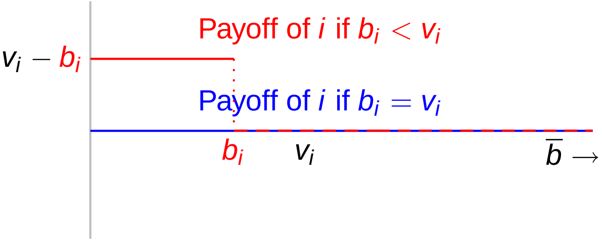

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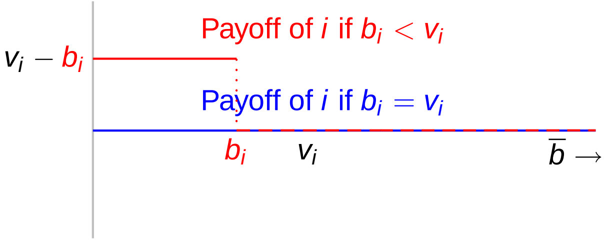
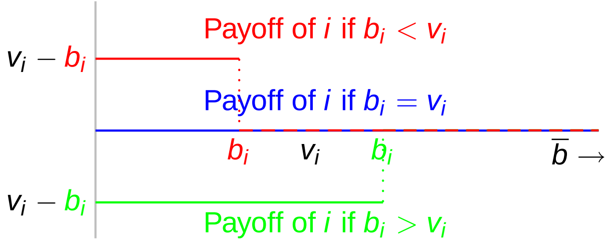
Analysis



$\Rightarrow$ bid of v_i *is weakly dominated by* bid of $b_i < v_i$

First-price sealed-bid auction

Analysis



Proposition

For player i with valuation v_i in a first-price sealed-bid auction, the bid v_i weakly dominates all higher bids but does **not** weakly dominate lower bids. In fact, the bid v_i is weakly dominated by any lower bid.

First-price sealed-bid auction

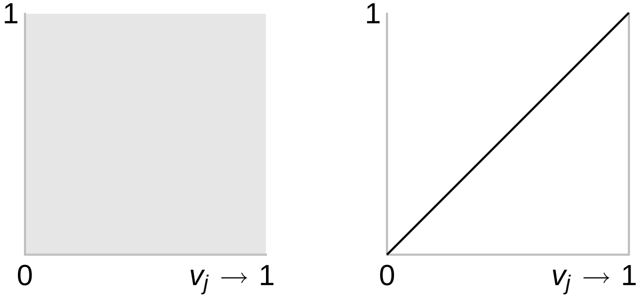
Example

- Two bidders

First-price sealed-bid auction

Example

- Two bidders
- Each bidder i believes that other bidder's valuation is distributed uniformly between 0 and 1, independently of v_i

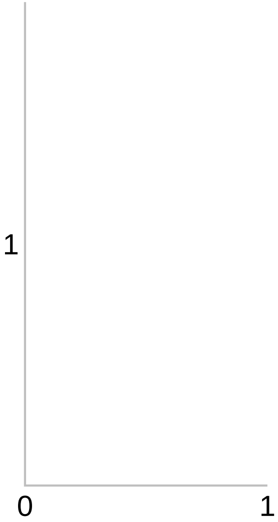


Probability density of v_j Cumulative distribution of v_j

First-price sealed-bid auction

Analysis of example

Claim The auction has a (symmetric) Nash equilibrium in which a player with valuation v bids $\frac{1}{2}v$. That is, each player bids exactly half her valuation.



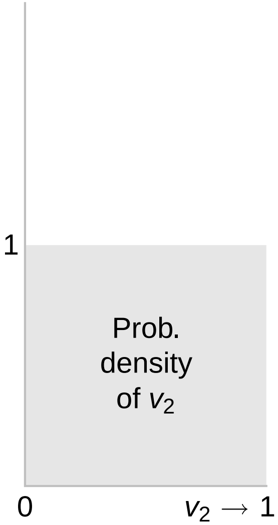
First-price sealed-bid auction

Analysis of example

Claim The auction has a (symmetric) Nash equilibrium in which a player with valuation v bids $\frac{1}{2}v$. That is, each player bids exactly half her valuation.

Argument

- Suppose that for each valuation v_2, bid of player 2 with valuation v_2 is $\frac{1}{2}v_2$



First-price sealed-bid auction

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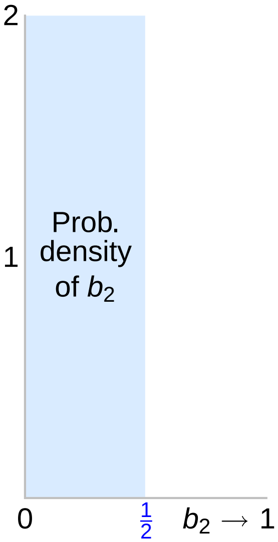
Argument

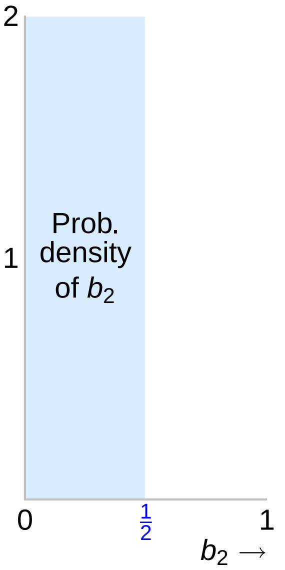
- Suppose that for each valuation v_2, bid of player 2 with valuation v_2 is $\frac{1}{2}v_2$
- Then player 1 believes that player 2's bids are distributed uniformly between 0 and $\frac{1}{2}$



First-price sealed-bid auction

Analysis of example

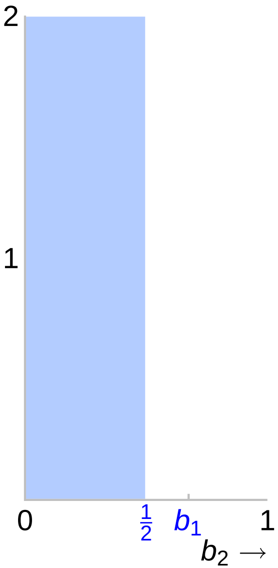
- Suppose player 1 bids b_1



First-price sealed-bid auction

Analysis of example

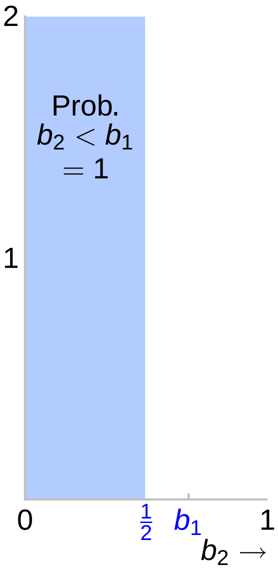
- Suppose player 1 bids b_1
- $b_1 > \frac{1}{2} \Rightarrow$



First-price sealed-bid auction

Analysis of example

- Suppose player 1 bids b_1
- $b_1 > \frac{1}{2} \Rightarrow$ player 1 surely wins



First-price sealed-bid auction

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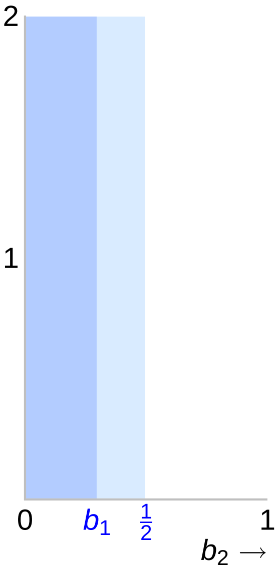
- Suppose player 1 bids b_1
- $b_1 > \frac{1}{2} \Rightarrow$ player 1 surely wins
- $b_1 \leq \frac{1}{2} \Rightarrow$ prob. player 1 wins $=$



First-price sealed-bid auction

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- Suppose player 1 bids b_1
- $b_1 > \frac{1}{2} \Rightarrow$ player 1 surely wins
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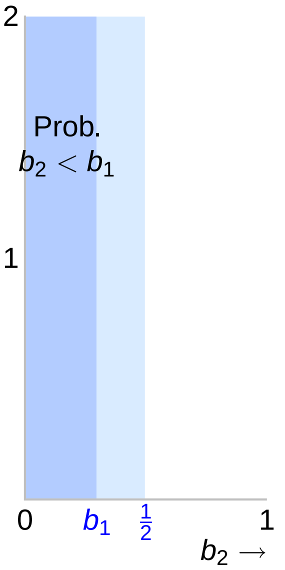

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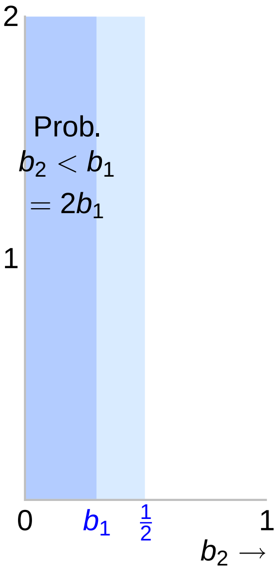

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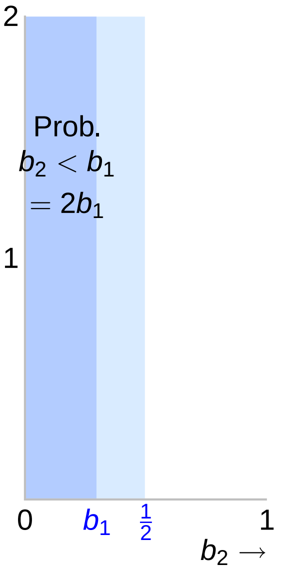
- So player 1's payoff is

$$\begin{cases} & \text{if } 0 \leq b_1 \leq \frac{1}{2} \\ & \text{if } b_1 > \frac{1}{2} \end{cases}$$



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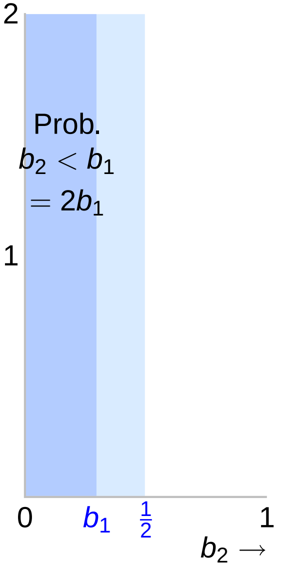
- er … off is

Prob. of winning

Payoff if wins

$$\begin{cases} 2b_1(v_1 - b_1) & \text{if } 0 \leq b_1 \leq \frac{1}{2} \\ & \text{if } b_1 > \frac{1}{2} \end{cases}$$



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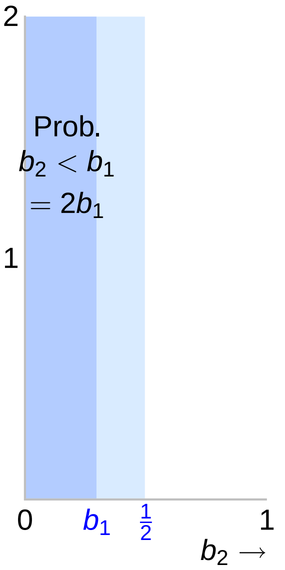

First-price sealed-bid auction

Analysis of example

- So far: if, for each valuation v_2, player 2's bid is $\frac{1}{2}v_2$, then expected payoff of player 1 with valuation v_1 to bidding b_1 is

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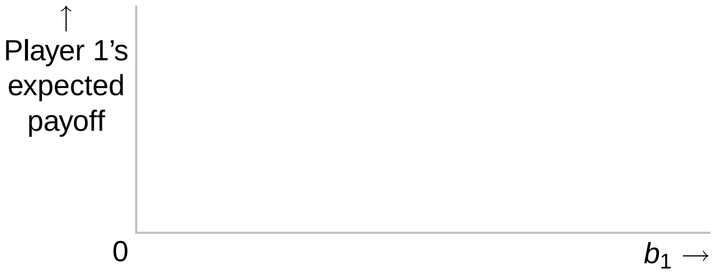

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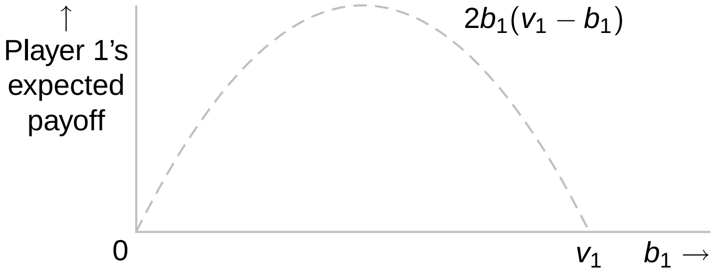

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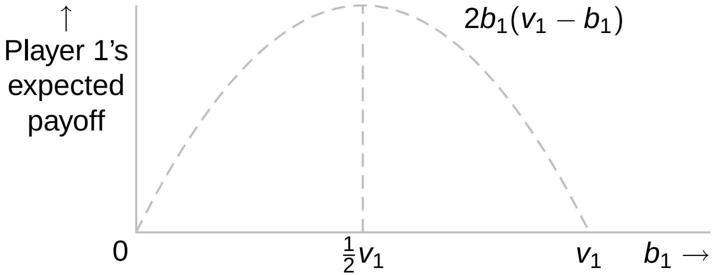

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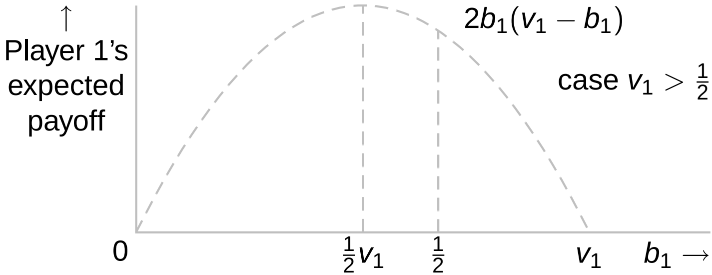

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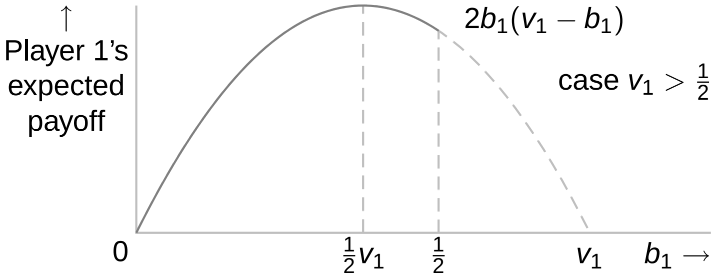

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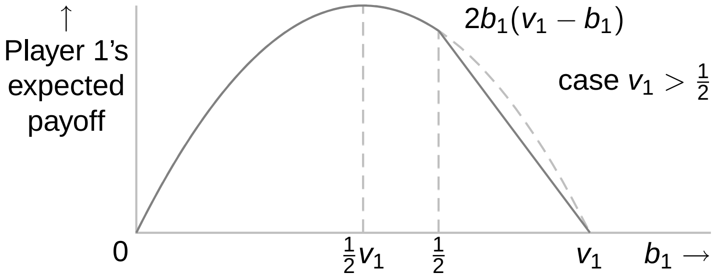

- Thus best response of player 1 with valuation v_1 is $\frac{1}{2}v_1$

First-price sealed-bid auction

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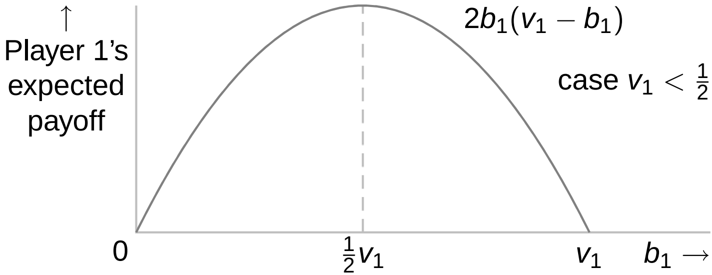

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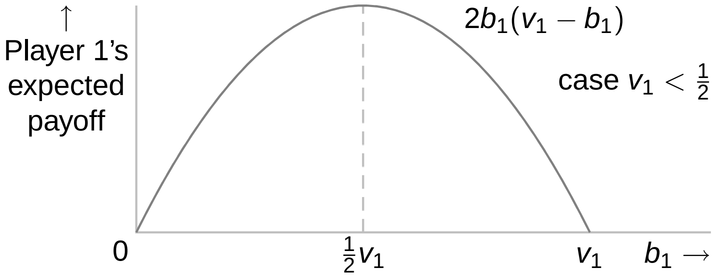

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First-price sealed-bid auction

Example: conclusion

- If, for each valuation v_2, player 2 with valuation v_2 bids $\frac{1}{2}v_2$, then best bid for player 1 with valuation v_1 is $\frac{1}{2}v_1$

First-price sealed-bid auction

Example: conclusion

- If, for each valuation v_2, player 2 with valuation v_2 bids $\frac{1}{2}v_2$, then best bid for player 1 with valuation v_1 is $\frac{1}{2}v_1$
- Argument for player 2 is symmetric: if, for each valuation v_1, player 1 with valuation v_1 bids $\frac{1}{2}v_1$ then best bid of player 2 with valuation v_2 is $\frac{1}{2}v_2$

First-price sealed-bid auction

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- In this equilibrium, as in equilibrium in second-price auction, object is sold to bidder with highest valuation

First-price sealed-bid auction

Example: interpretation of equilibrium bids

- When player i has valuation v_i, she wins if and only if player j's valuation v_j is less than v_i: $0 \leq v_j \leq v_i$

First-price sealed-bid auction

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First-price sealed-bid auction

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- When player i has valuation v_i, she wins if and only if player j's valuation v_j is less than v_i: $0 \leq v_j \leq v_i$
- Over all cases in which player j's valuation is less than v_i, expected value of player j's valuation is $\frac{1}{2}v_i$
- Thus equilibrium bid of player i with valuation v_i is expected value of player j's valuation conditional on that valuation being less than v_i: $\mathrm{E}(v_j \mid v_j < v_i)$

First-price sealed-bid auction

Generalizing the example

- Suppose there are n bidders

First-price sealed-bid auction

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- Choose $n - 1$ valuations randomly and independently (remember that every bidder believes every other bidder's valuation is drawn independently from same distribution)

First-price sealed-bid auction

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- The highest of these $n-1$ valuations is a random variable: its value depends on the $n-1$ valuations that were chosen

First-price sealed-bid auction

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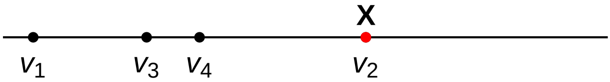
- Denote the highest of the $n - 1$ valuations by **X**



First-price sealed-bid auction

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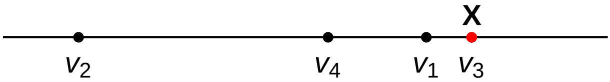

First-price sealed-bid auction

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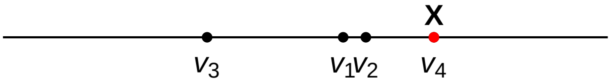

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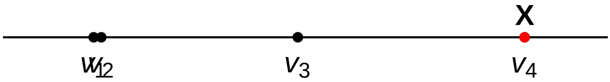

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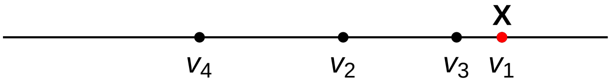

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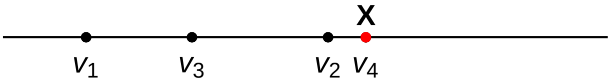

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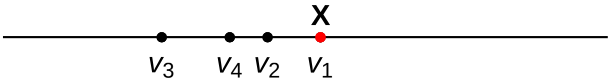

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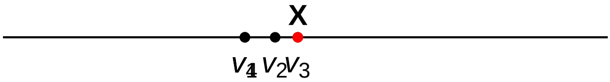

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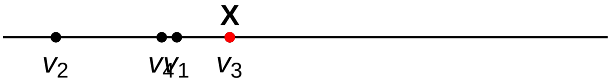

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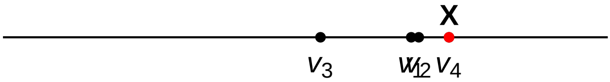

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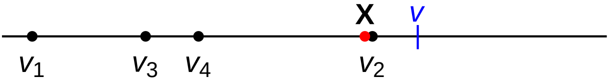

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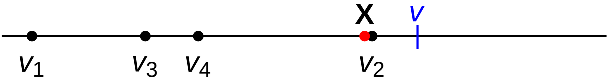

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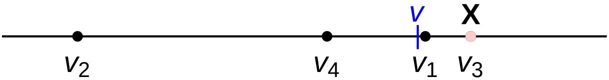

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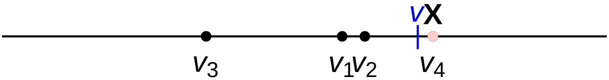

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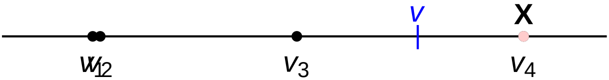

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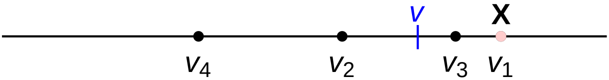

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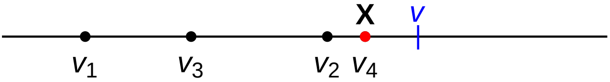

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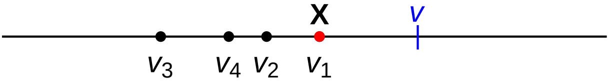

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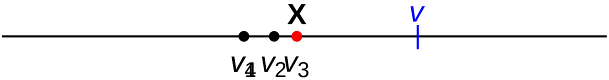

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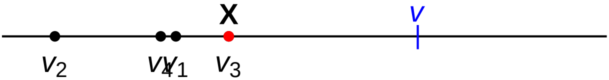

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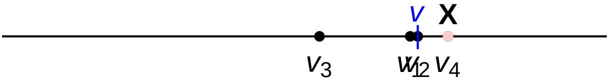

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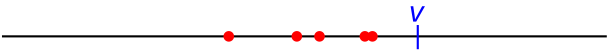

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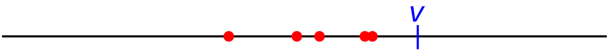

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First-price sealed-bid auction

Analogy: birthdays

- Birthdays (month, day) in group of people are more or less uniformly distributed from January 1 to December 31

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First-price sealed-bid auction

Generalizing the example

Proposition

For any number of bidders and any distribution of valuations, a first-price sealed-bid auction has a Nash equilibrium in which a player with valuation v bids $\mathrm{E}(\mathbf{X} \mid \mathbf{X} < v)$, the expected value of the highest of the other players' valuations conditional on v being higher than all the other valuations.

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First-price auction

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Proposition (*Revenue equivalence*)

Under our assumptions, first- and second-price auctions yield the same expected revenue

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Note: assumptions include risk-neutrality (payoffs $v - p$)

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Revenue equivalence

- Expected revenue of seller is same in first- and second-price auctions

Common value auctions

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- Interdependence of values introduces considerations not present when values are independent

Common value auctions

Drilling for oil

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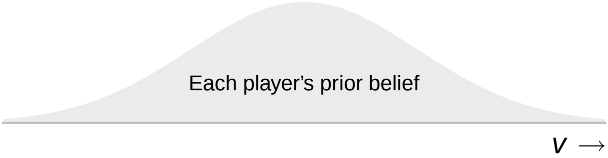
- Samples differ, so estimates of amount available based on samples differ

Common value auctions

Drilling for oil



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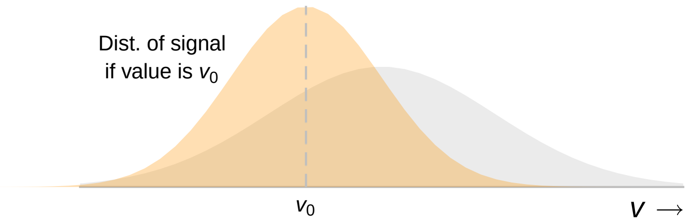

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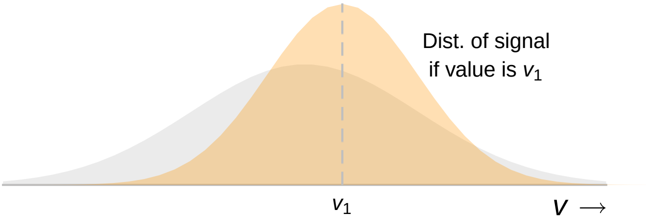

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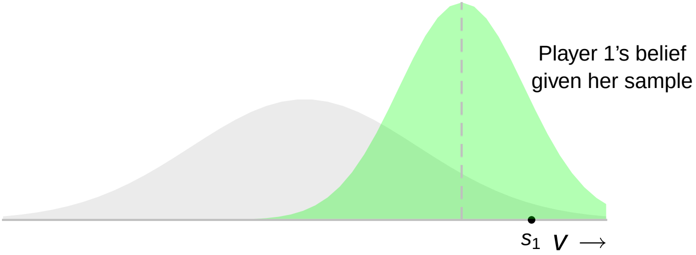

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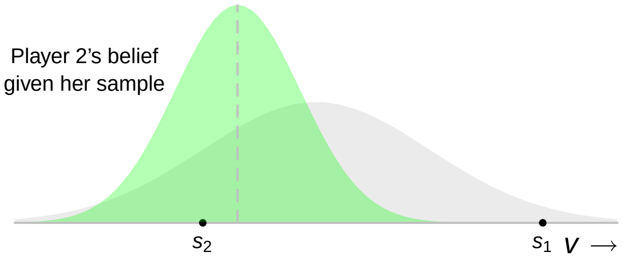

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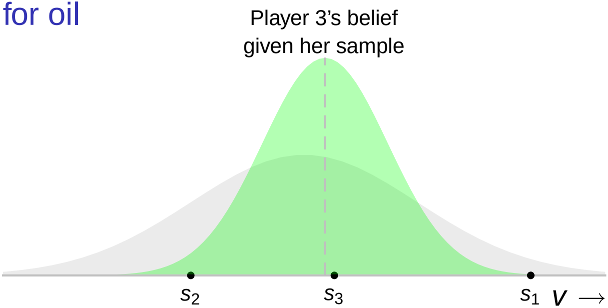
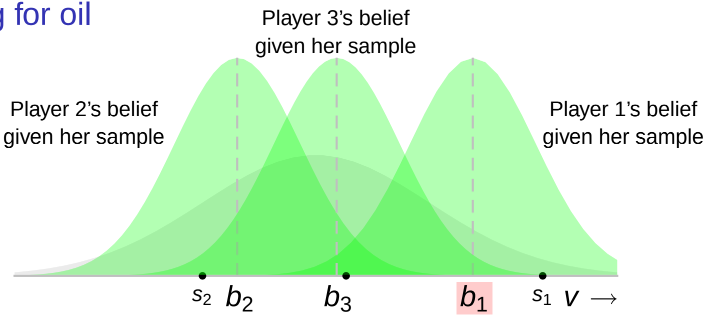
Drilling for oil



- Each player sees only her *own* sample
- On basis of sample and prior belief, forms probabilistic estimate of value
- Different players get different samples and form different estimates

Common value auctions

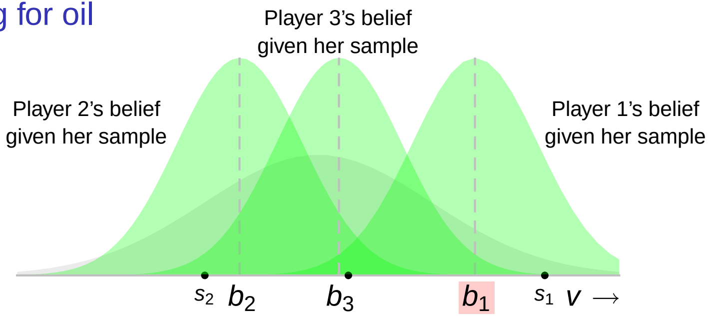
Drilling for oil



- Each player sees only her *own* sample
- On basis of sample and prior belief, forms probabilistic estimate of value
- Different players get different samples and form different estimates
- Suppose each player bids according to her own estimate

Common value auctions

Drilling for oil



- Highest bid wins $\Rightarrow$ player with best sample wins

Common value auctions

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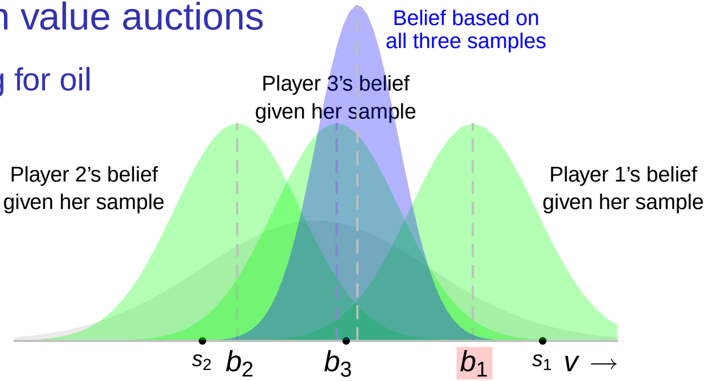

- Highest bid wins $\Rightarrow$ player with best sample wins
- *Best* sample typically overestimates value $\Rightarrow$ winner incurs loss

Common value auctions

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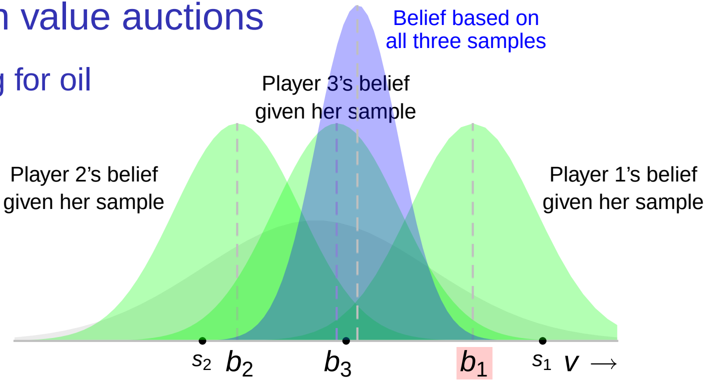

- Highest bid wins $\Rightarrow$ player with best sample wins
- *Best* sample typically overestimates value $\Rightarrow$ winner incurs loss
- Effect is known as winner's curse

Common value auctions

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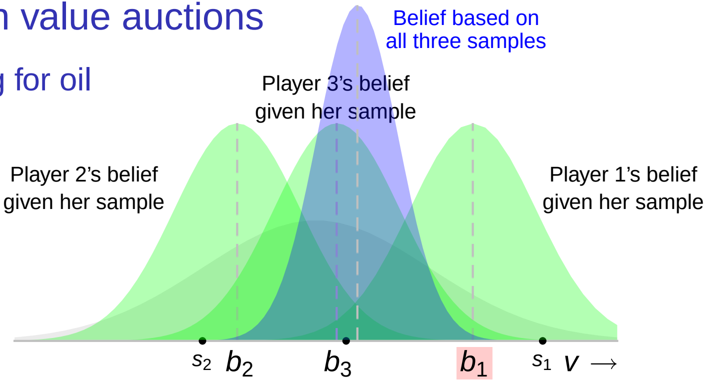

- If player wins, she knows other players' estimates of value are lower than hers

Common value auctions

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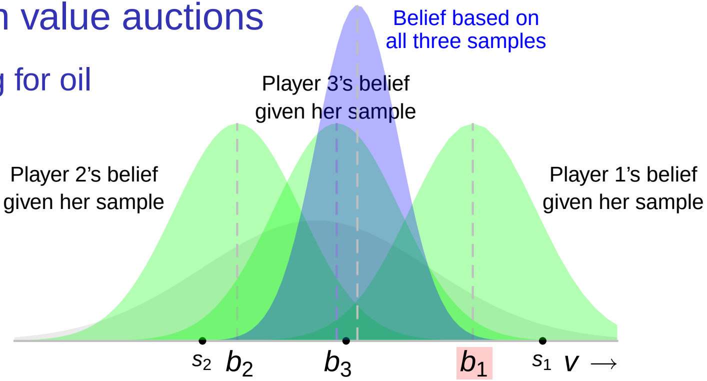

- If player wins, she knows other players' estimates of value are lower than hers
- She should taken this information into account, and base her bid on estimate of value *conditional on winning* (given other players' strategies)

Common value auctions

Getting your roof fixed

- You get three estimates to fix your roof

Common value auctions

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Common value auctions

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Common value auctions

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- You choose the lowest estimate—the third one, based on an underestimate of the difficulty of the job

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- The winner makes a loss

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- You choose the lowest estimate—the third one, based on an underestimate of the difficulty of the job
- The winner makes a loss
- The lesson: estimates should take into account that you will choose the *lowest* one

Common value auctions

Summary

- Even for second-price rule, not an equilibrium in common value auction for each player to bid her valuation based on her own information

Common value auctions

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Common value auctions

Summary

- Even for second-price rule, not an equilibrium in common value auction for each player to bid her valuation based on her own information
- Players who do so suffer “winner’s curse”
- Instead, player should base her bid on her estimate of her valuation of the object over all cases in which all other bids are lower than hers, given the other players’ equilibrium strategies