ECO316: Applied game theory

Lecture 6

Martin J. Osborne

Department of Economics
University of Toronto

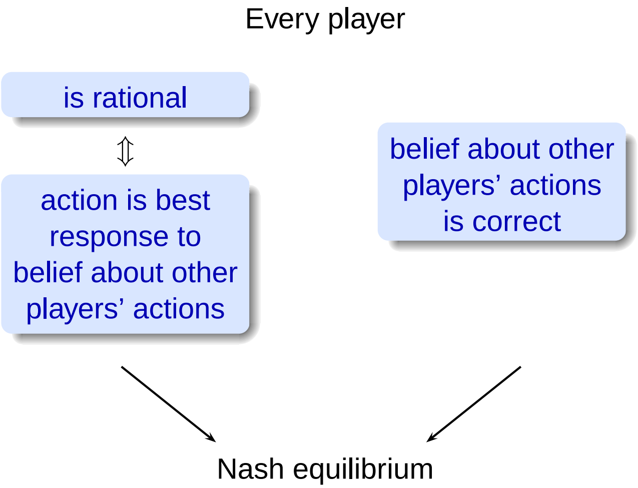
2017.10.12

Midterm

Date	Friday, October 20, 1:10pm to 3pm
Location	EX 100
Coverage	All the material in weeks 1–6
Weight	40%
Sample	Previous midterm exams, with solutions, are available on the course website

Table of contents

Implications of rationality



Every player

is rational

$\Updownarrow$

action is best response to belief about other players' actions

belief about other players' actions is correct

Nash equilibrium

Implications of rationality

Every player

is rational

⇕

action is best response to belief about other players' actions

believes that other players are rational

beli
pla

. . . and believes that other players believe she is rational

. . . and believes that other players believe she believes they are rational

. . . and so on

Which actions are best responses to some belief?

?

Beliefs

Definition

A belief of a player in a strategic game about the other players' actions is a probability distribution over the set of lists of the other players' actions.

Example

	q	$1 - q$
	H	L
H	$2, 2$	$0, 3$
L	$3, 0$	$1, 1$

- A belief of player 1 is a probability distribution over $\{H, L\}$
- The probability the belief assigns to each action is the probability with which player 1 believes player 2 will choose that action

Beliefs

Example

Players Three individuals

Actions Each player can choose A or B

Preferences Anything

- A belief of player 1 is a probability distribution over the set $\{(A,A),(A,B),(B,A),(B,B)\}$ of the pairs of actions of the other two players
- For example, one belief assigns
  - probability 0.25 to (A,A)
  - probability 0.25 to (A,B)
  - probability 0.5 to (B,A)
  - probability 0 to (B,B)

Best responses to a belief

In a strategic game, denote player *i*'s expected payoff when she uses the mixed strategy α_i and the other players' actions are a_{-i} by

$$U_i(\alpha_i, a_{-i})$$

Example

		H	L
(p)	H	$2, 2$	$0, 3$
$(1-p)$	L	$3, 0$	$1, 1$

$$U_1((p, 1-p), H) = 2p + 3(1-p)$$

$$U_1((p, 1-p), L) = 1 - p$$

Best responses to a belief

Definition

The mixed strategy α_i of player i is a best response to i's belief β_i if α_i maximizes player i's expected payoff

$$\sum_{a_{-i} \in A_{-i}} \beta_i(a_{-i}) U_i(\alpha_i, a_{-i}).$$

That is,

$$\sum_{a_{-i} \in A_{-i}} \beta_i(a_{-i}) U_i(\alpha_i, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} \beta_i(a_{-i}) U_i(\alpha'_i, a_{-i})$$

for every other mixed strategy α'_i of player i.

Rationality

Definition

A player is rational if her mixed strategy is a best response to *some* belief.

- How can we determine whether a mixed strategy is a best response to some belief?
- Let's start with a simpler question: when is a player's *action* a best response to a list of *actions* of the other players?

Strictly dominated actions

Example

Other players' actions

i's actions

	a'_{-i}	a''_{-i}	a'''_{-i}
T	1	4	0
B	2	6	3

Player *i*'s payoffs

- Is T a best response to any actions of the other players?
- No! Whatever actions i the other players take, B yields higher payoff than does T
- We say B *strictly dominates* T, or T is *strictly dominated* by B

Strictly dominated actions

Definition

Player i's action a_i'' strictly dominates her action a_i' if

a_i'' is better for i than a_i' *whatever* the other players do

or, precisely,

$u_i(a_i'', a_{-i}) > u_i(a_i', a_{-i})$ for *every* list a_{-i} of other players' actions

Strict domination

- If a''_i strictly dominates a'_i then a'_i is not best response to *any* list of the other players' actions
- Example

Other players' actions

i's actions

	a'_{-i}	a''_{-i}	a'''_{-i}
T	1	4	0
B	2	6	3

Player *i*'s payoffs

T is not a best response to any list of other players' actions

Strict domination

- In fact, if a''_i strictly dominates a'_i then a'_i is not a best response to any *probability distribution* over lists of the other players' actions — that is, to any belief
- Example

		Other players' actions		
		q_1	q_2	q_3
		a'_{-i}	a''_{-i}	a'''_{-i}
i's actions	T	1	4	0
	B	2	6	3

Player *i*'s payoffs

T is not a best response to any belief (q_1, q_2, q_3): for every (q_1, q_2, q_3),

$$2 \cdot q_1 + 6 \cdot q_2 + 3 \cdot q_3 > 1 \cdot q_1 + 4 \cdot q_2 + 0 \cdot q_3$$

Strict domination and best responses

Proposition

An action of player i in a strategic game that is strictly dominated by another action is not a best response to any belief of player i about the other players' actions. That is, an action for player i that is strictly dominated by another action is not a rational action for player i.

actions strictly dominated by another action

actions that are not rational

- The result does not say that an action that is *not* strictly dominated is necessarily rational
- In fact, we will see later than some actions that are not strictly dominated are *not* rational

Example: *Prisoner's Dilemma*

	Q	F
Q	2, 2	0, 3
F	3, 0	1, 1

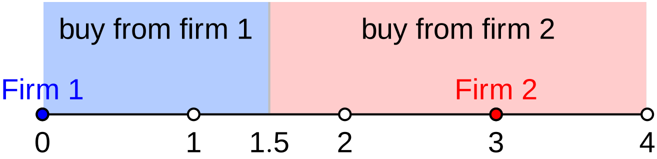
- Q is worse than F for player 1 if player 2 chooses Q
- Q is worse than F for player 1 if player 2 chooses F
- So Q is strictly dominated by F
- Rationality of player 1 $\Rightarrow$ she chooses F
- Same argument for player 2, so players' rationality alone implies outcome (F, F)

Examples of strict domination

Hotelling's game

- Consumers uniformly distributed along line segment $[0, 4]$
- Two firms choose positions on line segment
- Firms restricted to points 0, 1, 2, 3, 4
- Each consumer buys a unit from nearest firm (prices fixed)
- Each firm's payoff $=$ its market share

Example: firm 1 at 0, firm 2 at 3 $\Rightarrow$



Examples of strict domination

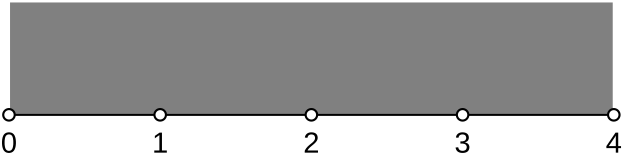
Hotelling's game

- Is any action strictly dominated?
- Compare actions 0 and 1 for firm 1

Firm 2

Firm 1		0	1	2	3	4
	0	2	0.5	1	1.5	2
	1	3.5	2	1.5	2	2.5

Firm 1's payoffs

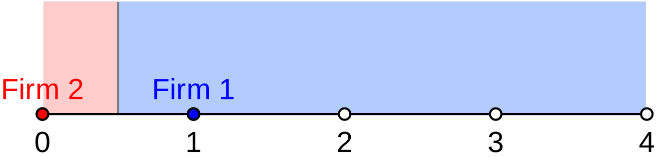


- 1 strictly dominates 0; symmetrically, 3 strictly dominates 4
- So no rational firm locates at 0 or 4

Examples of strict domination

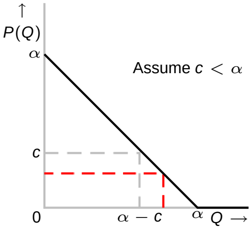
Hotelling's game

- Is action 1 strictly dominated?



- No: if other firm is at 0, 1 is best place to be!
- Similarly locations 2 and 3 are not strictly dominated
- So only strictly dominated locations are 0 and 4
- Hence rationality of each firm $\Rightarrow$ firm does not choose extreme locations (0 and 4)

Examples of strict domination



Assume $c < \alpha$

Cournot's duopoly game

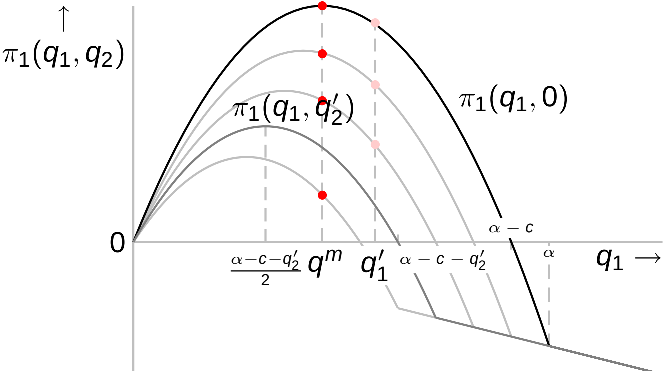
- Consider example with linear inverse demand $P(Q) = \alpha - Q$ and constant unit cost $c > 0$
- Is any output strictly dominated?
- Firm 1's profit:

	any q_2
$q_1 = 0$	0
$q_1 > \alpha - c$	?

- Every $q_1 > \alpha - c$ is strictly dominated by $q_1 = 0$
- Are any outputs smaller than $\alpha - c$ strictly dominated?

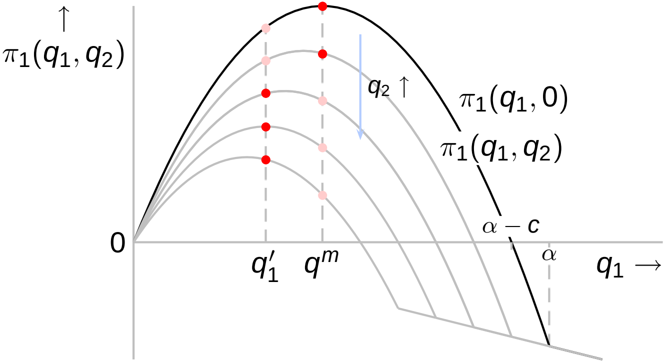
Cournot's duopoly game

- Does it ever make sense for a firm to produce more than the monopoly output ($q^m = \frac{1}{2}(\alpha - c)$)?
- Fix $q_1' > q^m$
- For *every* value of $q_2 \geq 0$, $\pi_1(q^m, q_2) > \pi_1(q_1', q_2)$
- So $q_1' > q^m$ is strictly dominated by q^m for firm 1



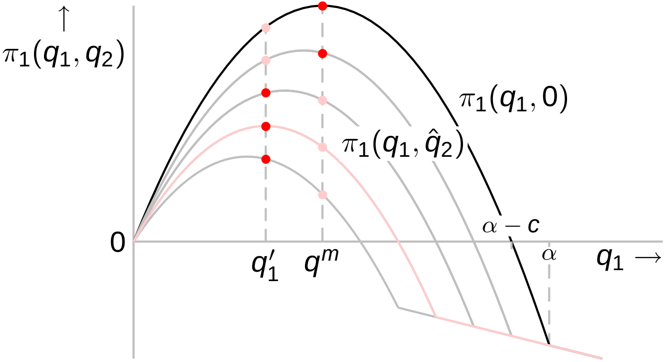
Cournot's duopoly game

- Now consider $q_1' < q^m$
- For q_2 small, $\pi(q^m, q_2) > \pi(q_1', q_2)$
- But for q_2 large enough, $\pi(q_1', q_2) > \pi(q^m, q_2)$
- So $q_1' < q^m$ is *not* strictly dominated by q^m



Cournot's duopoly game

- Is q_1' strictly dominated by *any* other output?
- Consider value of q_2 for which q_1' is firm 1's optimal output
- This value is $\hat{q}_2 = \alpha - c - 2q_1'$ [problem]
- $\pi_1(q_1', \hat{q}_2) > \pi_1(q_1, \hat{q}_2)$ for *all* $q_1 \neq q_1'$
- So $q_1' < q^m$ is not strictly dominated by *any* q_1



Cournot’s duopoly game

Conclusion

- Every output $> q^m$ is strictly dominated by q^m
- No output $< q^m$ is strictly dominated by q^m
- In fact, no output $< q^m$ is strictly dominated by *any* output
- So rationality of each firm $\Rightarrow$ each firm produces at most monopoly output

Rationality and strict domination

- Rationality ⇔ player's action is best response to *some* belief
- Earlier result:

Proposition

An action of player i in a strategic game that is strictly dominated by another action is not a best response to any belief of player i about the other players' actions.

- Are any *other* actions not best responses to any belief?

Never-best responses

	L (q)	R $(1-q)$	Exp. payoff
(0) T	1	1	1
$(\frac{1}{2})$ M	4	0	$4q$
$(\frac{1}{2})$ B	0	4	$4(1-q)$

Player 1's payoffs

- Is T a best response to any belief of player 1?
  - If $q < \frac{3}{4}$, payoff to $B >$ payoff to T
    if $q > \frac{1}{4}$, payoff to $M >$ payoff to T
  - So T is not a best response to any belief
- Is T strictly dominated by another action?
  - No: B yields lower payoff if player 2 chooses L and M yields lower payoff if player 2 chooses R
  - So rational player 1 does not use T, but T is not strictly dominated

Never-best responses

	L (q)	R $(1-q)$	Exp. payoff
(0) T	1	1	1
$(\frac{1}{2})$ M	4	0	$4q$
$(\frac{1}{2})$ B	0	4	$4(1-q)$

Player 1's payoffs

- Suppose player 1 uses mixed strategy $(0, \frac{1}{2}, \frac{1}{2})$:
  - expected payoff if player 2 uses L: $\frac{1}{2} \cdot 4 = 2$
  - expected payoff if player 2 uses R: $\frac{1}{2} \cdot 4 = 2$
- So mixed strategy $(0, \frac{1}{2}, \frac{1}{2})$ is better for player 1 than T regardless of player 2's action
- That, is, the *mixed strategy* $(0, \frac{1}{2}, \frac{1}{2})$ strictly dominates T
  - (Other mixed strategies also strictly dominate T)

Never-best responses and strict domination

Definition

Player i's mixed strategy α_i strictly dominates her action a_i if

i's expected payoff to $(\alpha_i, a_{-i}) >$ i's payoff to (a_i, a_{-i})

for *every* list a_{-i} of the other players' actions.

Necessary and sufficient condition for an action to be a best response to a belief:

Proposition

An action is not a best response to any belief if and only if it is strictly dominated by a mixed strategy

Never-best responses and strict domination

Summary

- Rationality $\Leftrightarrow$ player chooses best response to *some* belief
- Action strictly dominated by another action $\Rightarrow$ action is not best response to any belief $\Rightarrow$ not used by rational player
- Usually straightforward to identify such actions
- Action strictly dominated by mixed strategy $\Leftrightarrow$ action is not best response to any belief $\Rightarrow$ not used by rational player
- Can be hard to identify such actions

Relation with Nash equilibrium

- Can a strictly dominated action be used in a Nash equilibrium?
- No! A strictly dominated action isn't a best response to any list of the other players' actions
- Is it possible for a player to assign positive probability to a never-best response in a mixed strategy Nash equilibrium?
- No! Every action used with positive probability in a mixed strategy equilibrium must be a best response to the other players' mixed strategies

Relation with Nash equilibrium

Proposition

An action that is not a best response to any belief is not used with positive probability in any mixed strategy Nash equilibrium.

- So when looking for a mixed strategy equilibrium, we can eliminate from consideration actions that are strictly dominated by mixed strategies
- However, determining whether an action is strictly dominated by a mixed strategy can be difficult
- Mostly we will consider only strict domination by an action

Rationality and equilibrium

Every player

is rational

⇕

action is best response to belief about other players’ actions

believes that other players are rational

. . . and believes that other players believe she is rational

. . . and believes that other players believe she believes they are rational

. . . and so on

?

Drawing conclusions from other players' rationality

Example

	L	*C*	*R*
T	0, 4	4, 0	2, 1
M	1, 0	3, 1	3, 2
B	0, 2	2, 3	1, 1

- Player 1 is rational $\Rightarrow$ does not choose *B* (strictly dominated by *M*)
- Player 2 believes player 1 is rational $\Rightarrow$ player 2 believes player 1 does not choose *B*
  $\Rightarrow$ if player 2 is rational she does not choose *C*
- Player 1 believes that player 2 is rational *and* that player 2 believes player 1 is rational $\Rightarrow$ player 1 believes player 2 believes player 1 does not choose *B* and that player 2 therefore does not choose *C*
  $\Rightarrow$ if player 1 is rational, she does not choose *T*
- In one more step . . . player 2 does not choose *L*
  $\Rightarrow$ only action pair that remains is (M, R)

Iterated elimination of strictly dominated actions

- Process of sequentially deleting strictly dominated actions: iterated elimination of strictly dominated actions
- At some stage, more than one action may be strictly dominated; surviving action profiles are independent of order of elimination
- *If* every player is rational
  and every player believes every other player is rational
  and every player believes that every other player believes that the other players are rational
  and so forth . . .

  then the action profile that the players choose *survives iterated elimination of strictly dominated actions*

Iterated elimination of strictly dominated actions

- Every Nash equilibrium survives iterated elimination of strictly dominated actions (IESDA) (why?)
- But in many games, many action profiles that are not Nash equilibria also survive IESDA
- Change one payoff in game we just looked at:

	L	C	R
T	$0, 4$	$4, 0$	$2, 1$
M	$1, 0$	$3, 1$	$3, 2$
B	$2, 2$	$2, 3$	$1, 1$

- Now *no* action of either player is strictly dominated
  $\Rightarrow$ *all* action profiles survive IESDA

Iterated elimination of strictly dominated actions

		Vizzini	
		Goblet *A*	Goblet *B*
Man in black	Poison in goblet *A*	$1, -10$	$-10, 1$
	Poison in goblet *B*	$-10, 1$	$1, -10$

- In fact, in a random game, it is likely that no action of any player is strictly dominated, so that all action profiles survive IESDA
- However, in several games of economic interest, only Nash equilibria survive IESDA

Hotelling's game

- For each player, extreme locations are strictly dominated
- Eliminate extreme locations ⇒ remaining locations 1, 2, 3
- By the same argument as before 1 and 3 are strictly dominated in the game that results

⇒ only action pair that survives iterated elimination of strictly dominated actions is $(2, 2)$: both firms choose the median position

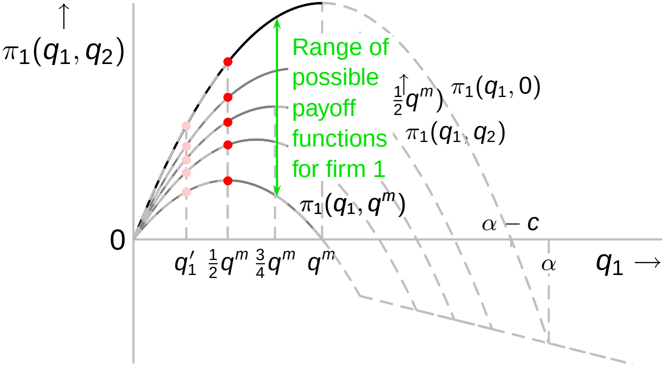
Hotelling's game

Summary

- Firms' rationality and their reasoning about each other's rationality $\Rightarrow$ outcome is $(2, 2)$, Nash equilibrium
- Argument generalizes to any finite number of possible positions for firms

Cournot's duopoly game

- We know that all outputs $> q^m$ $(= \frac{1}{2}(\alpha - c))$ are strictly dominated
- So eliminate all outputs $> q^m$ for firm 2
- Now any output $q_1' < \frac{1}{2}q^m$ for firm 1 is strictly dominated by $\frac{1}{2}q^m$



Cournot's duopoly game

Conclusion so far

- Every output $> q^m$ is strictly dominated (by q^m)
  $\Rightarrow$ no rational firm chooses output $> q^m$
- After outputs $> q^m$ are eliminated, every output $< \frac{1}{2}q^m$ is strictly dominated
  $\Rightarrow$ a firm that believes the other firm is rational does not rationally choose an output $< \frac{1}{2}q^m$
- After outputs $< \frac{1}{2}q^m$ are eliminated, every output $> \frac{3}{4}q^m$ is strictly dominated
  $\Rightarrow$ a firm that believes the other firm is rational does not rationally choose an output $< \frac{1}{2}q^m$ or $> \frac{3}{4}q^m$

Cournot's duopoly game

Continuing the process

- The process continues … until only the Nash equilibrium $(\frac{1}{3}(\alpha - c), \frac{1}{3}(\alpha - c))$ remains

Eventual conclusion

- The only action pair that survives iterated elimination of strictly dominated actions in the example of Cournot's duopoly game is the Nash equilibrium

Weakly dominated actions

	Q	*F*
Q	2, 2	0, 3
F	3, 0	1, 1

Prisoner's Dilemma

- In *Prisoner's Dilemma*, *F* strictly dominates *Q*

	Split	*Steal*
Split	£50,000, £50,000	0, £100,000
Steal	£100,000, 0	0, 0

Split or steal?

- In *Split or steal*, *Steal* does *not* strictly dominate *Split*
  - but *Steal* is never worse than *Split*
  - and is better if the other player chooses *Split*
- We say *Steal* weakly dominates *Split*

Weak domination

Definition

Player i's action a_i'' weakly dominates her action a_i' if

a_i'' is at least as good for i as a_i' for *all* actions of the other players

and

a_i'' is better for i than a_i' for *some* actions of the other players.

Precisely,

$u_i(a_i'', a_{-i}) \geq u_i(a_i', a_{-i})$ for *every* list a_{-i} of other players' actions
$u_i(a_i'', a_{-i}) > u_i(a_i', a_{-i})$ for *some* list a_{-i} of other players' actions

Weak domination

Example

Lists of other players’ actions

		a'_{-i}	a''_{-i}	a'''_{-i}
Player i	T	1	6	0
	B	2	6	3

Player i’s payoffs

The action B weakly dominates T:

- for every list of actions of the other players, B is at least as good as T for player i
- for the list of actions a'_{-i} (and also for a'''_{-i}), B is better than T for player i

Weak domination and Nash equilibrium

- Can a weakly dominated action be used in a Nash equilibrium?

Player 2

		L	R
Player 1	T	$1, 1$	$0, 0$
	B	$0, 0$	$0, 0$

- (B, R) is a Nash equilibrium
- B is weakly dominated by T (and R is weakly dominated by L)
- So a weakly dominated action *can* be used in a Nash equilibrium

Weak domination

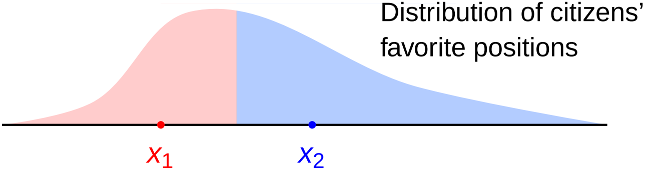
- Case against choosing *weakly* dominated action isn’t as strong as case against choosing *strictly* dominated action
- In *Split or steal?*, for example, *Split* is optimal if player is *certain* that other player will choose *Steal* (and *Steal* is also optimal in this case)
- But any doubt, however small, makes *Split* suboptimal
- There is no *positive* incentive to choose a weakly dominated action

	Split	*Steal*
Split	£50,000, £50,000	0, £100,000
Steal	£100,000, 0	0, 0

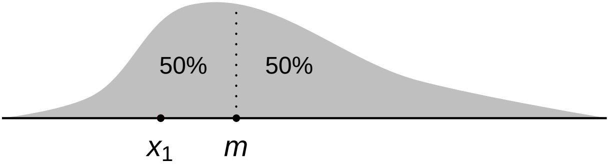
Split or steal?

Hotelling's model of electoral competition

- Two parties
- Each party chooses a position on the line
- Large number of citizens
- Each citizen has favorite position
- Each citizen votes for party whose position is closest to her favorite position
- Party who gets most votes wins
- Each party prefers to win than to tie than to lose



Hotelling's model of electoral competition



- Is any position weakly dominated?
- Let $x_1 < m$ (median favorite position)

Party 2

		$x_2 < x_1$	$x_2 = x_1$	$x_1 < x_2 < m$	$x_2 = m$	$x_2 > m$
Party 1	x_1	win	tie	lose	lose	l, t, or w
	m	win	win	win	tie	win

Outcomes for party 1

- So m weakly dominates *every* $x_1 < m$
- Similarly m weakly dominates every $x_1 > m$

Hotelling’s model of electoral competition

Conclusion

- For each player, m weakly dominates every $x < m$ and every $x > m$
- … so m weakly dominates *every* other action
- m is not weakly dominated
- Thus m is the only action that is not weakly dominated

Bertrand's model of duopoly

- Two firms
- Each firm chooses a price
- Total demand at price p is $D(p)$
- If $p_i < p_j$ then firm i gets all the demand and firm j gets no demand
- If $p_i = p_j = p$ then each firm gets the demand $\frac{1}{2}D(p)$
- Payoff of firm i

$$\begin{cases} (p_i - c)D(p_i) & \text{if } p_i < p_j \\ \frac{1}{2}(p_i - c)D(p_i) & \text{if } p_i = p_j \\ 0 & \text{if } p_i > p_j \end{cases}$$

Bertrand's model of duopoly

- Is any price weakly dominated?
- Let $p_1 < c$

		Firm 2	
		$p_2 < p_1$	$p_2 \geq p_1$
Firm 1	$p_1 < c$	0	< 0
	c	0	0

Payoffs for firm 1

- So c weakly dominates any $p_1 < c$

Bertrand's model of duopoly

- Let $p_1 > c$ with $D(p_1) > 0$

		Firm 2	
		$p_2 < p_1$	$p_2 \geq p_1$
Firm 1	c	0	0
	$p_1 > c$	0	> 0

Payoffs for firm 1

- So any $p_1 > c$ weakly dominates c!

$\Rightarrow$ in Nash equilibrium, both firms use weakly dominated action!

Bertrand's model of duopoly

Summary

- Every $p < c$ is weakly dominated by c
- The price c is weakly dominated by every $p > c$ for which $D(p) > 0$
- Is any price $> c$ weakly dominated? [Problem Set 6]

Bertrand's model of duopoly

Version with discrete prices

- Suppose that a price has to be an integral number of cents
- Then two Nash equilibria, (c, c) and $(c + 1, c + 1)$ [Problem Set 2]
- Is $p = c + 1$ weakly dominated? [Problem Set 6]

Strict domination and rationality: summary

- A player is defined to be rational if the action she chooses is a best response to some belief
- An action that is strictly dominated by another action is not a best response to any belief
- An action is a best response to some belief if and only if it is not strictly dominated by a *mixed* strategy
- If every player is rational, and believes that the other players are rational, and believes that every other player believes that the other players are rational . . . then the action profile survives *iterated* elimination of strictly dominated actions
- In many games, a large number of action profiles survive iterated elimination of strictly dominated actions
- But in some key economic models, the Nash equilibrium is the only action profile that survives

Weak domination and rationality: summary

- Using a weakly dominated action is not ruled out by rationality
- But is optimal only for beliefs that assign probability one to specific actions
- A weakly dominated action is not optimal for beliefs that assign positive probability to every list of actions of the other players
- In many economic models, restricting players to actions that are not weakly dominated makes sense

Domination and Nash equilibrium: summary

- A strictly dominated action is not used with positive probability in any mixed strategy Nash equilibrium
- A weakly dominated action may be used in a Nash equilibrium