ECO316: Applied game theory

Lecture 4

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Strategic game

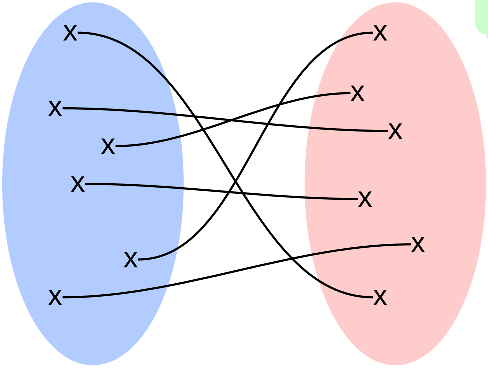
	L	R
T	$1, 0$	$0, 4$
B	$0, 1$	$2, 0$

- Nash equilibrium?
- Game has no Nash equilibrium
- What happens if game is played?

Games without Nash equilibria: Steady state

	L	*R*
T	$1, 0$	$0, 4$
B	$0, 1$	$2, 0$

Members of populations are randomly matched



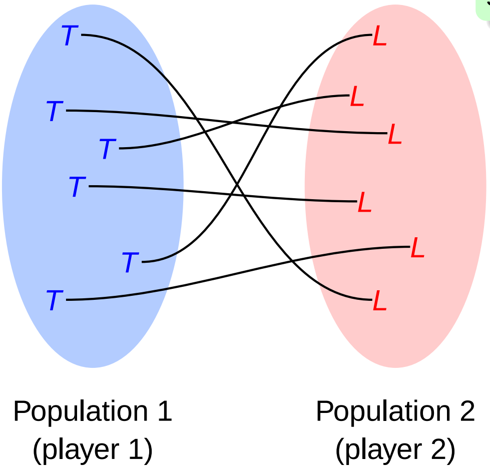
Population 1 (player 1)

Population 2 (player 2)

Games without Nash equilibria: Steady state

	L	*R*
T	1, 0	0, 4
B	0, 1	2, 0

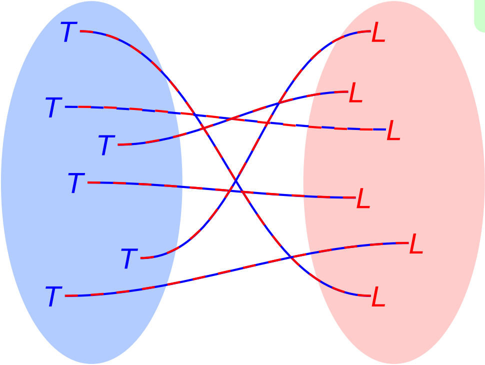
Is this pattern of behavior a steady state?



Games without Nash equilibria: Steady state

	L	*R*
T	1, 0	0, 4
B	0, 1	2, 0

Not steady state: player 2's want to switch to *R*



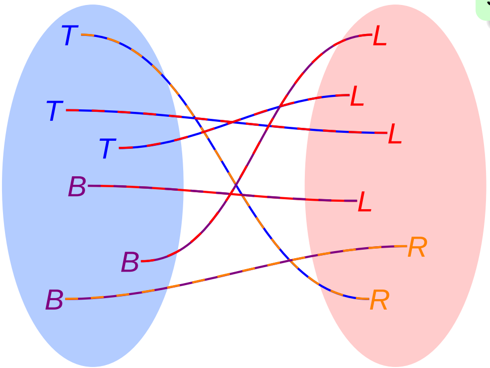
Population 1 (player 1)

Population 2 (player 2)

Games without Nash equilibria: Steady state

	L	*R*
T	1, 0	0, 4
B	0, 1	2, 0

Is this pattern of behavior a steady state?



Population 1 (player 1)

Population 2 (player 2)

Games without Nash equilibria

	(q) L	$(1-q)$ R
T	$1,0$	$0,4$
B	$0,1$	$2,0$

- Given frequencies of L and R for player 2, what is optimal action for player 1?
- Expected payoffs of player 1:

$$T : q \cdot 1 + (1-q) \cdot 0 = q$$
$$B : q \cdot 0 + (1-q) \cdot 2 = 2(1-q)$$

$\Rightarrow$ T is better if $q > 2(1-q)$, or $q > \frac{2}{3}$

$\Rightarrow$ B is better if $q < \frac{2}{3}$

$\Rightarrow$ If $q = \frac{2}{3}$ then expected payoffs to T and B are equal $\Rightarrow$ player 1 is indifferent between T and B

Games without Nash equilibria

	L	R
$(p)\ T$	1, 0	0, 4
$(1-p)\ B$	0, 1	2, 0

- Given frequencies of T and B for player 1, what is optimal action for player 2?
- Expected payoffs of player 2:

$$L : p \cdot 0 + (1-p) \cdot 1 = 1 - p$$
$$R : p \cdot 4 + (1-p) \cdot 0 = 4p$$

$\Rightarrow$ L is better if $1 - p > 4p$, or $p < \frac{1}{5}$

$\Rightarrow$ R is better if $p > \frac{1}{5}$

$\Rightarrow$ If $p = \frac{1}{5}$ then expected payoffs to L and R are equal $\Rightarrow$ player 2 is indifferent between L and R

Games without Nash equilibria

	(q) L	$(1-q)$ R
(p) T	$1, 0$	$0, 4$
$(1-p)$ B	$0, 1$	$2, 0$

Conclusion

- both players are indifferent between their two actions if and only if $p = \frac{1}{5}$ and $q = \frac{2}{3}$

$\Rightarrow$ in a steady state in which some player 1's choose T and some choose B and some player 2's choose L and some choose R, we must have $p = \frac{1}{5}$ and $q = \frac{2}{3}$

- $p = \frac{1}{5}$, $q = \frac{2}{3}$ is consistent with *stochastic steady state*
- $((\frac{1}{5}, \frac{4}{5}), (\frac{2}{3}, \frac{1}{3}))$ is a *mixed strategy Nash equilibrium*

Games without Nash equilibria

Equilibrium payoffs

	$(\frac{2}{3})$ L	$(\frac{1}{3})$ R
$(\frac{1}{5})$ T	$1, 0$	$0, 4$
$(\frac{4}{5})$ B	$0, 1$	$2, 0$

- Player 1's expected payoffs:

  to T: $\frac{2}{3} \cdot 1 + \frac{1}{3} \cdot 0 = \frac{2}{3}$ to B: $\frac{2}{3} \cdot 0 + \frac{1}{3} \cdot 2 = \frac{2}{3}$

  $\Rightarrow$ player 1's expected payoff in equilibrium is $\frac{2}{3}$
- Player 2's expected payoffs:

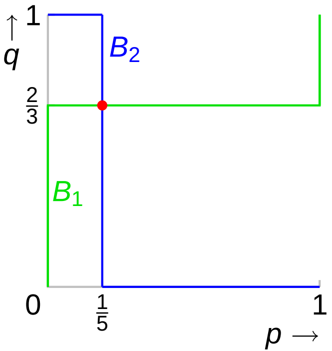
  to L: $\frac{1}{5} \cdot 0 + \frac{4}{5} \cdot 1 = \frac{4}{5}$ to R: $\frac{1}{5} \cdot 4 + \frac{4}{5} \cdot 0 = \frac{4}{5}$

  $\Rightarrow$ player 2's expected payoff in equilibrium is $\frac{4}{5}$

Games without Nash equilibria

Constructing best response functions

	L (q)	R $(1-q)$
T (p)	$1, 0$	$0, 4$
B $(1-p)$	$0, 1$	$2, 0$



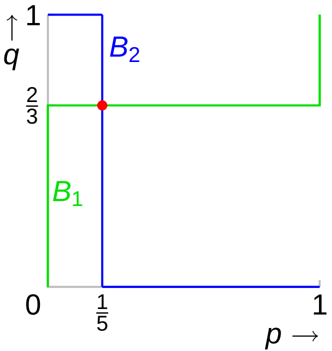
Player 1

- B $(p = 0)$ is better than T $(p = 1)$ if $q < \frac{2}{3}$
  $\Rightarrow$ P1's best response to any $q < \frac{2}{3}$ is $p = 0$ (i.e. B)
- T is better than B if $q > \frac{2}{3}$ $\Rightarrow$ best response $p = 1$
- T and B equally good if $q = \frac{2}{3}$ $\Rightarrow$ *every* p is best response

Games without Nash equilibria

Constructing best response functions

	L (q)	R $(1-q)$
T (p)	$1, 0$	$0, 4$
B $(1-p)$	$0, 1$	$2, 0$



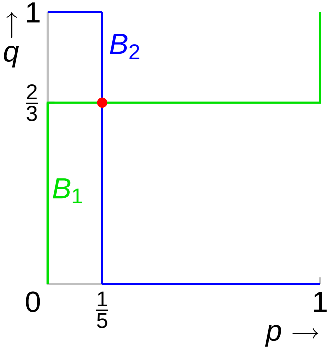
Player 2

- L $(q = 1)$ is better than R $(q = 0)$ if $p < \frac{1}{5}$
  $\Rightarrow$ P2's best response to any $p < \frac{1}{5}$ is $q = 1$ (i.e. L)
- R is better than L if $p > \frac{1}{5}$ $\Rightarrow$ best response $q = 0$
- L and R equally good if $p = \frac{1}{5}$ $\Rightarrow$ *every* q is best response

Games without Nash equilibria

Constructing best response functions

	L (q)	R $(1-q)$
T (p)	$1, 0$	$0, 4$
B $(1-p)$	$0, 1$	$2, 0$



Equilibrium

- Intersection of B_1 and B_2 corresponds to mixed strategy Nash equilibrium:

$$((p, 1-p), (q, 1-q)) = \left(\left(\tfrac{1}{5}, \tfrac{4}{5}\right), \left(\tfrac{2}{3}, \tfrac{1}{3}\right)\right)$$

Games without Nash equilibria

Summary

	L (q)	R $(1-q)$
T (p)	$1, 0$	$0, 4$
B $(1-p)$	$0, 1$	$2, 0$

- Game has no (“pure strategy”) Nash equilibrium
- But it has a unique mixed strategy Nash equilibrium, in which player 1’s mixed strategy is $(\frac{1}{5}, \frac{4}{5})$ and player 2’s mixed strategy is $(\frac{2}{3}, \frac{1}{3})$
- The equilibrium corresponds to a stochastic steady state

Mixed strategy Nash equilibrium

Two interpretations

1. For each player in game, large population of people who may play the role of that player. Equilibrium probability of an action $=$ fraction of population that uses that action.
2. Each player individually randomizes between her actions. That is, she uses a mixed strategy—a probability distribution over her actions.

- Model fits both interpretations
- First interpretation often fits an application
- But I will usually use language of second interpretation, which is more convenient
- Will call a mixed strategy that assigns probability 1 to a single action a pure strategy

Mixed strategy Nash equilibrium

Mixed strategy

A mixed strategy for a player is a probability distribution over the player's set of actions

Mixed strategy Nash equilibrium

A mixed strategy Nash equilibrium of a strategic game is a profile of mixed strategies with the property that no player can increase her (expected) payoff by choosing a different mixed strategy, *given* the other players' mixed strategies

Mixed strategy Nash equilibrium

Expected payoffs

- Previously, payoffs were purely ordinal:

$$u(a) = 0, u(b) = 1, u(c) = 4$$
$$v(a) = 0, v(b) = 3, v(c) = 4$$

$\Rightarrow$ u and v represent *same* preferences over a, b, and c

- But now that payoff numbers are used to calculate expected payoffs, they are not purely ordinal:

for u: a with prob. $\frac{1}{2}$ & c with prob. $\frac{1}{2} \Rightarrow$ expected payoff 2
$\Rightarrow$ *better than b*
for v: a with prob. $\frac{1}{2}$ & c with prob. $\frac{1}{2} \Rightarrow$ expected payoff 2
$\Rightarrow$ *worse than b*

- $\Rightarrow$ expected payoffs represent *different* preferences

Mixed strategy Nash equilibrium

Proposition

The expected values of the payoff functions u and v represent same preferences over lotteries if and only if for some $\alpha > 0$ and β,

$$v(x) = \alpha u(x) + \beta \quad \text{for all } x.$$

Example

$$u(a) = 0, u(b) = 1, u(c) = 4$$
$$v(a) = 0, v(b) = 3, v(c) = 4$$

Can we find $\alpha > 0$ and β to satisfy condition in Proposition? We need

$$\begin{array}{lllll} x = a: & v(a) = \alpha u(a) + \beta & \Rightarrow & 0 = \alpha \cdot 0 + \beta & \Rightarrow \beta = 0 \\ x = b: & v(b) = \alpha u(b) + \beta & \Rightarrow & 3 = \alpha \cdot 1 + \beta & \Rightarrow \alpha = 3 \\ x = c: & v(c) = \alpha u(c) + \beta & \Rightarrow & 4 = \alpha \cdot 4 + \beta & \Rightarrow \alpha = 1 \end{array}$$

Impossible! So u and v represent different preferences

Mixed strategy Nash equilibrium

Proposition

The expected values of the payoff functions u and v represent same preferences over lotteries if and only if for some $\alpha > 0$ and β,

$$v(x) = \alpha u(x) + \beta \quad \text{for all } x.$$

Another example

$$u(a) = 0, u(b) = 1, u(c) = 4$$
$$v(a) = 1, v(b) = 3, v(c) = 9$$

Can we find $\alpha > 0$ and β to satisfy condition in Proposition?

We need

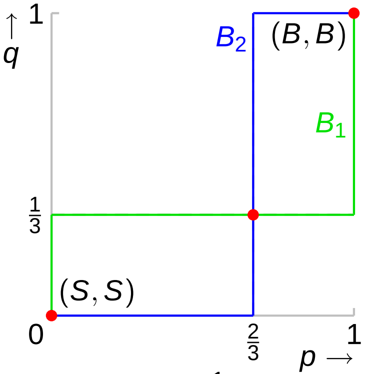
$$\begin{aligned} x = a: &\quad 1 = \alpha \cdot 0 + \beta &\quad \Rightarrow \beta = 1 \\ x = b: &\quad 3 = \alpha \cdot 1 + \beta &\quad \Rightarrow \alpha = 2 \\ x = c: &\quad 9 = \alpha \cdot 4 + \beta &\quad \Rightarrow \alpha = 2 \end{aligned}$$

So $\alpha = 2$ and $\beta = 1$ satisfy Proposition. So u and v represent same preferences.

Mixed strategy Nash equilibrium

Example: BoS

	B (q)	S $(1-q)$
B (p)	$2, 1$	$0, 0$
S $(1-p)$	$0, 0$	$1, 2$

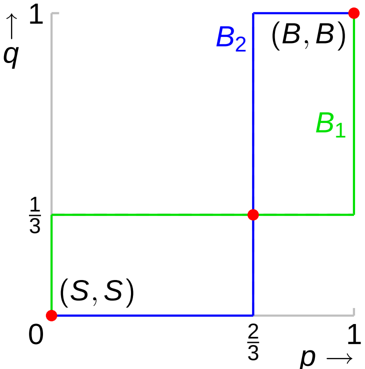


- For P1: B better than S if $2q > 1 - q$, or $q > \frac{1}{3}$
- S better than B if $q < \frac{1}{3}$
- B and S have same expected payoff if $q = \frac{1}{3}$

Mixed strategy Nash equilibrium

Example: BoS

	B (q)	S $(1-q)$
B (p)	$2, 1$	$0, 0$
S $(1-p)$	$0, 0$	$1, 2$

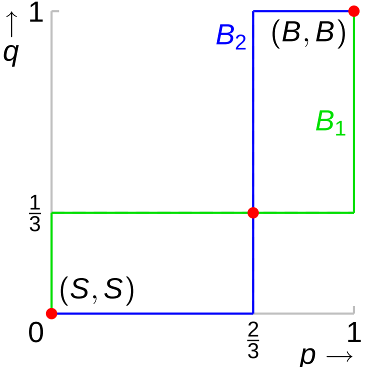


- For P2: B better than S if $p > 2(1-p)$, or $p > \frac{2}{3}$
- S better than B if $p < \frac{2}{3}$
- B and S have same expected payoff if $p = \frac{2}{3}$

Mixed strategy Nash equilibrium

Example: BoS

	B (q)	S $(1-q)$
B (p)	$2, 1$	$0, 0$
S $(1-p)$	$0, 0$	$1, 2$

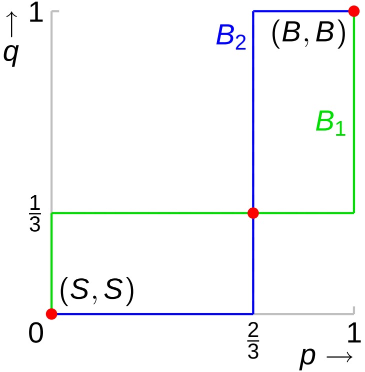


- 3 mixed strategy Nash equilibria: $((0,1),(0,1))$, $((\frac{2}{3},\frac{1}{3}),(\frac{1}{3},\frac{2}{3}))$, and $((1,0),(1,0))$
- $((0,1),(0,1))$ and $((1,0),(1,0))$ are pure strategy equilibria $((S,S)$ and $(B,B))$

Mixed strategy Nash equilibrium

Example: BoS

	B (q)	S $(1-q)$
B (p)	$2, 1$	$0, 0$
S $(1-p)$	$0, 0$	$1, 2$



Equilibrium payoffs:

$$\text{equilibrium } ((0,1),(0,1)) : (1,2)$$
$$\text{equilibrium } ((\tfrac{2}{3},\tfrac{1}{3}),(\tfrac{1}{3},\tfrac{2}{3})) : (\tfrac{2}{3},\tfrac{2}{3})$$
$$\text{equilibrium } ((1,0),(1,0)) : (2,1)$$

Mixed strategy Nash equilibrium

Mixed and pure equilibria

- No randomization allowed $\Rightarrow$ BoS has two Nash equilibria, (B, B) and (S, S)
- Randomization allowed $\Rightarrow$ these two equilibria survive (as $((1, 0), (1, 0))$ and $((0, 1), (0, 1)))$ and a third one $(((\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3})))$ appears
- In any game,
  - a Nash equilibrium when no randomization is allowed survives as a mixed strategy Nash equilibrium when randomization is allowed
  - a mixed strategy Nash equilibrium in which each player assigns probability 1 to a single action remains a Nash equilibrium when no randomization is allowed

Mixed strategy Nash equilibrium

- In every example, a mixed strategy Nash equilibrium exists even if a pure strategy equilibrium does not exist
- Same is true for any game in which each player has *finitely* many actions
- Main result in John Nash's PhD thesis

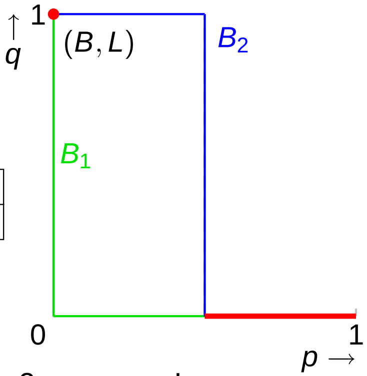
Proposition

Every strategic game in which every player has finitely many actions has a mixed strategy Nash equilibrium

Mixed strategy Nash equilibrium

Example

	L (q)	R $(1-q)$
T (p)	$0, 1$	$0, 2$
B $(1-p)$	$2, 2$	$0, 1$

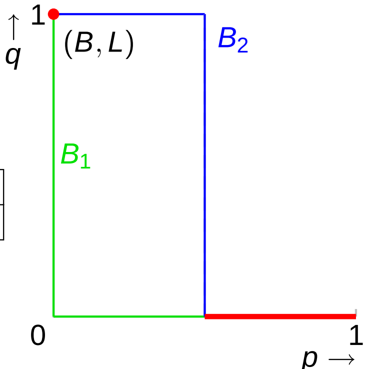


- For P1: T better than B if $0 > 2q$, or never!
- B better than T if $0 < 2q$, or $q > 0$
- B and T have same expected payoff if $q = 0$

Mixed strategy Nash equilibrium

Example

	L (q)	R $(1-q)$
T (p)	$0, 1$	$0, 2$
B $(1-p)$	$2, 2$	$0, 1$

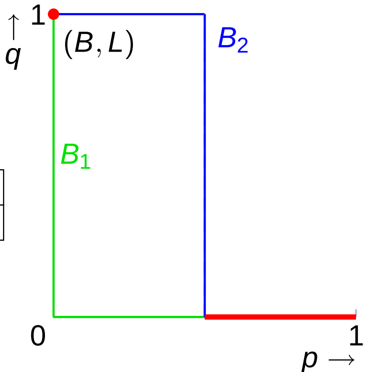


- For P2: L better than R if $p + 2(1-p) > 2p + 1 - p$, or $p < \frac{1}{2}$
- R better than L if $p > \frac{1}{2}$
- L and R have same expected payoff if $p = \frac{1}{2}$

Mixed strategy Nash equilibrium

Example

	L (q)	R $(1-q)$
T (p)	$0, 1$	$0, 2$
B $(1-p)$	$2, 2$	$0, 1$



- Infinitely many mixed strategy Nash equilibria:
  - $((0,1),(1,0))$
  - any strategy pair $((p,1-p),(0,1))$ for $\frac{1}{2} \leq p \leq 1$

Mixed strategy Nash equilibrium

Example

		$\frac{2}{3}$ L	$\frac{1}{3}$ R	P1's exp. payoff
$\frac{1}{5}$	T	$1, 0$	$0, 4$	$\frac{2}{3}$
$\frac{4}{5}$	B	$0, 1$	$2, 0$	$\frac{2}{3}$
0	X	$0, 0$	$0, 0$	0

Mixed strategy Nash equilibrium?

- Ignoring X, game is one we saw before
- In equilibrium of game excluding X, player 1's expected payoff is $\frac{2}{3}$
- Player 1's payoff to X against player 2's strategy $(\frac{2}{3}, \frac{1}{3})$ is 0
- So strategy $(\frac{1}{5}, \frac{4}{5}, 0)$ of player 1 is optimal in whole game
- Hence $((\frac{1}{5}, \frac{4}{5}, 0), (\frac{2}{3}, \frac{1}{3}))$ is a mixed strategy Nash equilibrium of the whole game

Mixed strategy Nash equilibrium

Example

		$\frac{2}{3}$ L	$\frac{1}{3}$ R	P1 exp. payoff
$\frac{1}{5}$	T	$1, 0$	$0, 4$	$\frac{2}{3}$
$\frac{4}{5}$	B	$0, 1$	$2, 0$	$\frac{2}{3}$
	X	$1, 0$	$1, 0$	1

Mixed strategy Nash equilibrium?

- Change payoff of player 1 to (X, L) and (X, R) to 1
- Is $((\frac{1}{5}, \frac{4}{5}, 0), (\frac{2}{3}, \frac{1}{3}))$ an equilibrium of this game?
- Player 1's payoff to X is now $1 > \frac{2}{3}$, so player 1 is better off choosing X than choosing T or B
- So $((\frac{1}{5}, \frac{4}{5}, 0), (\frac{2}{3}, \frac{1}{3}))$ is not an equilibrium (Problem Set asks you to find equilibria)

Mixed strategy Nash equilibrium

In general,

- player must be indifferent between actions to be willing to randomize

so

1. expected payoff to every action to which equilibrium mixed strategy assigns positive probability must be same
2. expected payoff to every action to which equilibrium mixed strategy assigns probability 0 must be less than or equal to expected payoffs to actions used with positive probability

Mixed strategy Nash equilibrium

Can use this fact to check whether strategy profile is mixed strategy Nash equilibrium

Example

	L (0)	C ($\frac{1}{3}$)	R ($\frac{2}{3}$)	
T ($\frac{3}{4}$)	$\cdot, 2$	$3, 3$	$1, 1$	$\frac{5}{3}$
M (0)	$\cdot, \cdot$	$0, \cdot$	$2, \cdot$	$\frac{4}{3}$
B ($\frac{1}{4}$)	$\cdot, 4$	$5, 1$	$0, 7$	$\frac{5}{3}$
	$\frac{5}{2}$	$\frac{5}{2}$	$\frac{5}{2}$	

(Unspecified payoffs are irrelevant.) Is indicated mixed strategy pair a Nash equilibrium?

- Calculate expected payoffs to P1's actions:
  - T: $0 \cdot ? + \frac{1}{3} \cdot 3 + \frac{2}{3} \cdot 1 = \frac{5}{3}$
  - M: $0 \cdot ? + \frac{1}{3} \cdot 0 + \frac{2}{3} \cdot 2 = \frac{4}{3}$
  - B: $0 \cdot ? + \frac{1}{3} \cdot 5 + \frac{2}{3} \cdot 0 = \frac{5}{3}$

Mixed strategy Nash equilibrium

Can use this fact to check whether strategy profile is mixed strategy Nash equilibrium

Example

	L (0)	C ($\frac{1}{3}$)	R ($\frac{2}{3}$)	
T ($\frac{3}{4}$)	$\cdot, 2$	$3, 3$	$1, 1$	$\frac{5}{3}$
M (0)	$\cdot, \cdot$	$0, \cdot$	$2, \cdot$	$\frac{4}{3}$
B ($\frac{1}{4}$)	$\cdot, 4$	$5, 1$	$0, 7$	$\frac{5}{3}$
	$\frac{5}{2}$	$\frac{5}{2}$	$\frac{5}{2}$	

(Unspecified payoffs are irrelevant.) Is indicated mixed strategy pair a Nash equilibrium?

- Calculate expected payoffs to P2's actions:
  - L: $\frac{3}{4} \cdot 2 + 0 \cdot ? + \frac{1}{4} \cdot 4 = \frac{5}{2}$
  - C: $\frac{3}{4} \cdot 3 + 0 \cdot ? + \frac{1}{4} \cdot 1 = \frac{5}{2}$
  - R: $\frac{3}{4} \cdot 1 + 0 \cdot ? + \frac{1}{4} \cdot 7 = \frac{5}{2}$

Mixed strategy Nash equilibrium

Can use this fact to check whether strategy profile is mixed strategy Nash equilibrium

Example

	L (0)	C ($\frac{1}{3}$)	R ($\frac{2}{3}$)	
T ($\frac{3}{4}$)	$\cdot, 2$	$3, 3$	$1, 1$	$\frac{5}{3}$
M (0)	$\cdot, \cdot$	$0, \cdot$	$2, \cdot$	$\frac{4}{3}$
B ($\frac{1}{4}$)	$\cdot, 4$	$5, 1$	$0, 7$	$\frac{5}{3}$
	$\frac{5}{2}$	$\frac{5}{2}$	$\frac{5}{2}$	

(Unspecified payoffs are irrelevant.) Is indicated mixed strategy pair a Nash equilibrium?

- Every action to which P1's mixed strategy assigns positive probability (T and B) yields same payoff
- Every action to which P1's mixed strategy assigns probability 0 (M) yields at most this payoff

Mixed strategy Nash equilibrium

Can use this fact to check whether strategy profile is mixed strategy Nash equilibrium

Example

	L (0)	C ($\frac{1}{3}$)	R ($\frac{2}{3}$)	
T ($\frac{3}{4}$)	$\cdot, 2$	$3, 3$	$1, 1$	$\frac{5}{3}$
M (0)	$\cdot, \cdot$	$0, \cdot$	$2, \cdot$	$\frac{4}{3}$
B ($\frac{1}{4}$)	$\cdot, 4$	$5, 1$	$0, 7$	$\frac{5}{3}$
	$\frac{5}{2}$	$\frac{5}{2}$	$\frac{5}{2}$	

(Unspecified payoffs are irrelevant.) Is indicated mixed strategy pair a Nash equilibrium?

- Every action to which P2's mixed strategy assigns positive probability (C and R) yields same payoff
- Every action to which P2's mixed strategy assigns probability 0 (L) yields at most this payoff

Mixed strategy Nash equilibrium

Can use this fact to check whether strategy profile is mixed strategy Nash equilibrium

Example

	L (0)	C ($\frac{1}{3}$)	R ($\frac{2}{3}$)	
T ($\frac{3}{4}$)	$\cdot, 2$	$3, 3$	$1, 1$	$\frac{5}{3}$
M (0)	$\cdot, \cdot$	$0, \cdot$	$2, \cdot$	$\frac{4}{3}$
B ($\frac{1}{4}$)	$\cdot, 4$	$5, 1$	$0, 7$	$\frac{5}{3}$
	$\frac{5}{2}$	$\frac{5}{2}$	$\frac{5}{2}$	

(Unspecified payoffs are irrelevant.) Is indicated mixed strategy pair a Nash equilibrium?

$\Rightarrow$ strategy pair is mixed strategy Nash equilibrium

Summary

- Mixed strategy equilibrium models stochastic steady state
- Every finite game has at least one mixed strategy equilibrium
- Pure strategy equilibria are special cases of mixed strategy equilibria
- In a mixed strategy equilibrium, every player
  - gets same expected payoff from every action to which she assigns positive probability
  - cannot get more payoff from action to which she assigns probability zero than from action to which she assigns positive probability