Economics 316

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Solutions for Problem Set 2

1. (a) Every profile $(e, \ldots, e)$, where e is an integer from 0 to K, is a Nash equilibrium. In the equilibrium $(e, \ldots, e)$, each player's payoff is e. The profile $(e, \ldots, e)$ is a Nash equilibrium because if player i chooses $e_i < e$ then her payoff is $2e_i - e_i = e_i < e$, and if she chooses $e_i > e$ then her payoff is $2e - e_i < e$.

   (b) Consider an action profile $(e_1, \ldots, e_n)$ in which not all effort levels are the same. Suppose that e_i is the minimum. Consider some player j whose effort level exceeds e_i. Her payoff is $2e_i - e_j < e_i$, while if she deviates to the effort level e_i her payoff is $2e_i - e_i = e_i$. Thus she can increase her payoff by deviating, so that $(e_1, \ldots, e_n)$ is not a Nash equilibrium.

2. The pair (c, c) of prices remains a Nash equilibrium. The argument is the same as before. The profit of each firm is zero for $p = (c, c)$; if a firm increases its price, its profit remains zero, and if it decreases its price, its profit is negative, given that demand is positive for $p_i < \overline{p}$ and $c < \overline{p}$.

   Further, as before, there is no other Nash equilibrium. The arguments for the cases in which at least one firm's price is at most c are the same as before. The remaining case requires only minor modification (because under the assumptions on D, a monopoly price may not exist). Suppose $p_i > c$, $p_j > c$, and $p_i \geq p_j$. Then if $D(p_j) > 0$, firm i can increase its profit by reducing its price slightly below p_j, and if $D(p_j) = 0$, it can increase its profit by reducing its price slightly below $\overline{p}$ (where demand is positive).

3. The set of Nash equilibria is the set of pairs $(p, p+1)$ where $101 \leq p \leq 200$.

   Every such pair is a Nash equilibrium by the following argument.

   If $101 \leq p \leq 200$ then at $(p, p+1)$ the payoff of firm 1 is $(p - 100)(\alpha - p)$, which is positive.

- If firm 1 reduces its price to p', its payoff changes to $(p' - 100)(\alpha - p')$. Now, the maximizer of the function $(x - 100)(\alpha - x)$ is $x = \frac{1}{2}(100 + \alpha)$, which exceeds 200 because $\alpha > 300$. So p and p' are both less than the maximizer; hence $(p' - 100)(\alpha - p') < (p - 100)(\alpha - p)$.
- If firm 1 increases its price to $p + 1$ its payoff changes from $(p - 100)(\alpha - p)$ to $\frac{1}{2}(p + 1 - 100)(\alpha - p - 1)$. Given $p \geq 101$, we have $p - 100 \geq \frac{1}{2}(p + 1 - 100)$, and $\alpha - p > \alpha - p - 1$, so firm 1's payoff decreases if it raises its price to $p + 1$.
- If firm 1 raises its price further, its payoff becomes zero.

If $101 \leq p \leq 200$ then at $(p, p + 1)$ the payoff of firm 2 is 0.

- If firm 2 reduces its price to p its payoff become $\frac{1}{2}(p - 200)(\alpha - p) \leq 0$.
- If firm 2 reduces its price further, below p, its payoff becomes $(p - 200)(\alpha - p) < 0$.
- If firm 2 raises its price, its payoff remains 0.

To show that the game has no other Nash equilibria, consider each pair of prices different from $(p, p + 1)$ for $101 \leq p \leq 200$. Denote the unit cost of firm i by c_i (so that $c_1 = 100$ and $c_2 = 200$), and the optimal price of a monopolist with cost c_i by p_i^m. Note that by the assumption that $\alpha > 300$ we have $p_1^m > 200$, and because $c_2 > c_1$ we have $p_2^m > p_1^m$. The pairs of prices other than $(p, p + 1)$ for $101 \leq p \leq 200$ can be divided into the following regions. (For each reason, I give one profitable deviation; in most cases there are also other profitable deviations.) The regions are illustrated in Figure 1.

- $p_1 \leq c_1 - 1$ and $p_1 \leq p_2$: firm 1 can profitably increase p_1 to c_1
- $p_2 \leq c_2 - 1$ and $p_2 \leq p_1$: firm 2 can profitably increase p_2 to c_2
- $p_1 = c_1$ and $p_2 \geq c_1 + 1$: firm 1 can profitably increase p_1 to $c_1 + 1$
- $p_1 \geq c_1 + 1$, $p_2 \geq p_1 + 2$ and $p_1 \leq p_1^m - 1$: firm 1 can profitably increase p_1 to $p_1 + 1$
- $p_2 \leq p_1$ and $c_2 \leq p_2 \leq p_1^m$: firm 1 can profitably reduce p_1 to $p_2 - 1$
- $p_1 \geq p_1^m$ and $p_2 \geq p_1 + 1$: firm 2 can profitably reduce p_2 to p_1
- $p_2 \leq p_1$ and $p_2 \geq p_1^m$: firm 1 can profitably reduce p_1 to p_1^m

4. Following the analysis in class, the best response function of firm 1 is

$$b_1(q_2) = \begin{cases} \frac{1}{2}(\alpha - c_1 - q_2) & \text{if } q_2 \le \alpha - c_1 \\ 0 & \text{otherwise} \end{cases}$$

while that of firm 2 is

$$b_2(q_1) = \begin{cases} \frac{1}{2}(\alpha - c_2 - q_1) & \text{if } q_1 \le \alpha - c_2 \\ 0 & \text{otherwise.} \end{cases}$$

To find the Nash equilibrium, first plot these two functions. Each function has the same general form as the best response function of either firm in the case studied in the text. However, the fact that $c_1 \neq c_2$ leads to two qualitatively different cases when we combine the two functions to find a Nash equilibrium. If c_1 and c_2 do not differ very much then the functions in the analogue of Figure 59.1 in the book intersect at a pair of outputs that are both positive. If c_1 and c_2 differ a lot, however, the functions intersect at a pair of outputs in which $q_1 = 0$.

Precisely, if $c_1 \le \frac{1}{2}(\alpha + c_2)$ then the downward-sloping parts of the best response functions intersect (as in Figure 59.1), and the game has a unique Nash equilibrium, given by the solution of the two equations

$$\begin{aligned} q_1 &= \tfrac{1}{2}(\alpha - c_1 - q_2) \\ q_2 &= \tfrac{1}{2}(\alpha - c_2 - q_1). \end{aligned}$$

This solution is

$$(q_1^*, q_2^*) = \left(\tfrac{1}{3}(\alpha - 2c_1 + c_2), \tfrac{1}{3}(\alpha - 2c_2 + c_1)\right).$$

If $c_1 > \frac{1}{2}(\alpha + c_2)$ then the downward-sloping part of firm 1's best response function lies below the downward-sloping part of firm 2's best response function (as in Figure 2), and the game has a unique Nash equilibrium, $(q_1^*, q_2^*) = (0, \frac{1}{2}(\alpha - c_2))$.

In summary, the game always has a unique Nash equilibrium, defined as follows:

$$\begin{cases} \left(\tfrac{1}{3}(\alpha - 2c_1 + c_2), \tfrac{1}{3}(\alpha - 2c_2 + c_1)\right) & \text{if } c_1 \le \frac{1}{2}(\alpha + c_2) \\ \left(0, \tfrac{1}{2}(\alpha - c_2)\right) & \text{if } c_1 > \frac{1}{2}(\alpha + c_2). \end{cases}$$

The output of firm 2 exceeds that of firm 1 in every equilibrium.

If c_2 decreases then firm 2's output increases and firm 1's output either falls, if $c_1 \le \frac{1}{2}(\alpha + c_2)$, or remains equal to 0, if $c_1 > \frac{1}{2}(\alpha + c_2)$. The total output increases and the price falls.

5. Player 1 chooses c_1 to maximize

$$c_1 + c_2 - (c_1 + c_2)^2 - (c_1)^2$$

given c_2. That is, player 1 chooses c_1 to maximize

$$c_1 + c_2 - 2c_1c_2 - (c_2)^2 - 2(c_1)^2.$$

The solution satisfies

$$1 - 2c_2 - 4c_1 = 0,$$

so that player 1's best response function is given by

$$b_1(c_2) = \tfrac{1}{4}(1 - 2c_2).$$

Symmetrically, player 2's best response function is given by

$$b_2(c_1) = \tfrac{1}{4}(1 - 2c_1).$$

An equilibrium (c_1^*, c_2^*) satisfies

$$c_1^* = b_1(c_2^*)$$
$$c_2^* = b_2(c_1^*).$$

Solving this system we find that

$$c_1^* = c_2^* = \tfrac{1}{6}.$$

Thus the game has a unique Nash equilibrium, $(c_1^*, c_2^*) = (\frac{1}{6}, \frac{1}{6})$.

6. Firm 1's payoff function is $p_1(10 - p_1 + 2p_2)$, so that its best response function is $b_1(p_2) = 5 + p_2$.

Firm 2's payoff function is $p_2(20 - p_2 + \frac{1}{2}p_1)$, so that its best response function is $b_2(p_1) = 10 + \frac{1}{4}p_1$.

A pair of prices (p_1^*, p_2^*) is a Nash equilibrium if and only if $b_1(p_2^*) = p_1^*$ and $b_2(p_1^*) = p_2^*$. Solving these two equations simultaneously yields $(p_1^*, p_2^*) = (20, 15)$. Thus the game has a unique Nash equilibrium, $(20, 15)$.

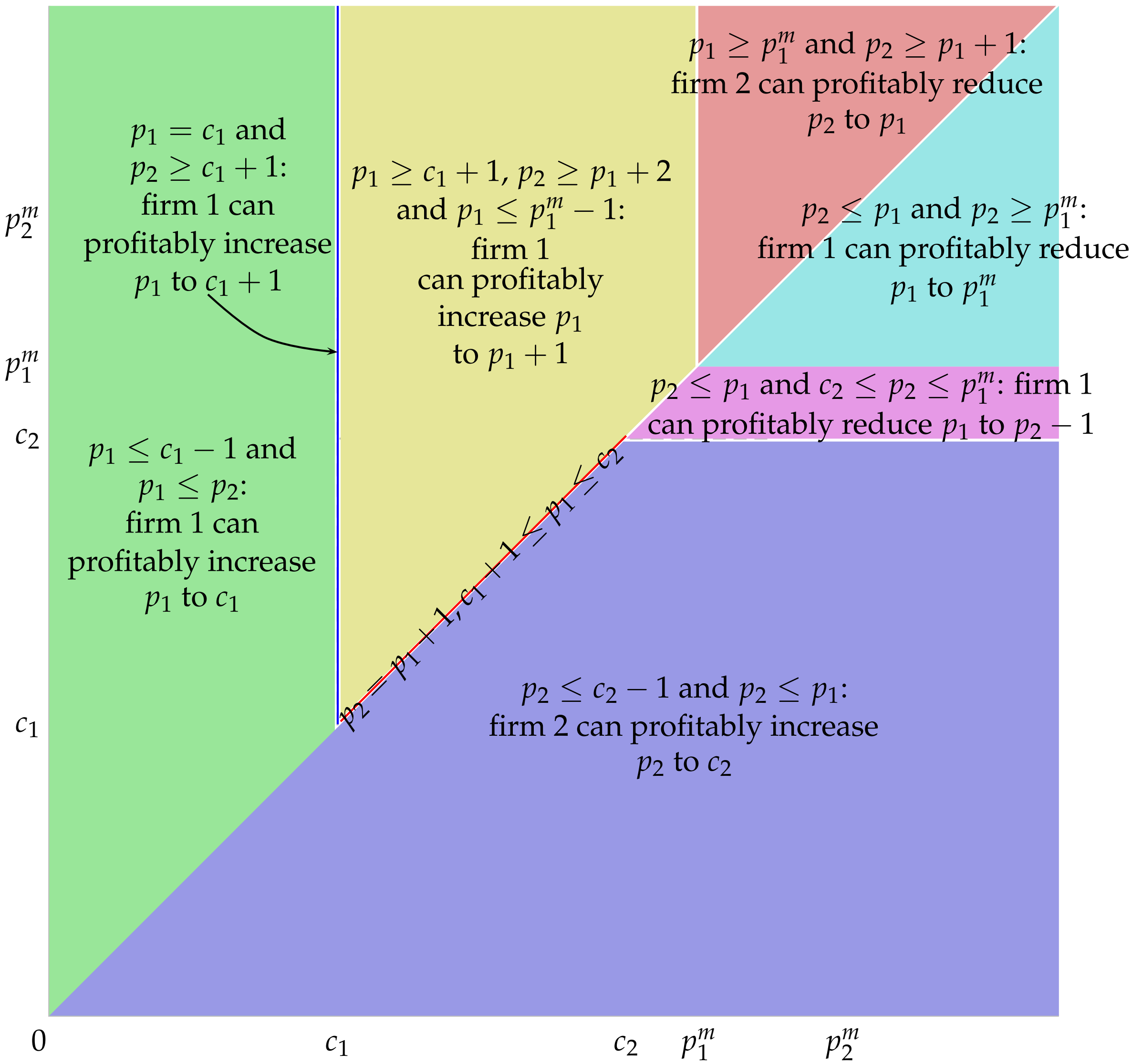

Figure 1. Possible price pairs in Problem 3. (The figure is drawn for $\alpha = 350$.) The Nash equilibrium price pairs are along the red line.

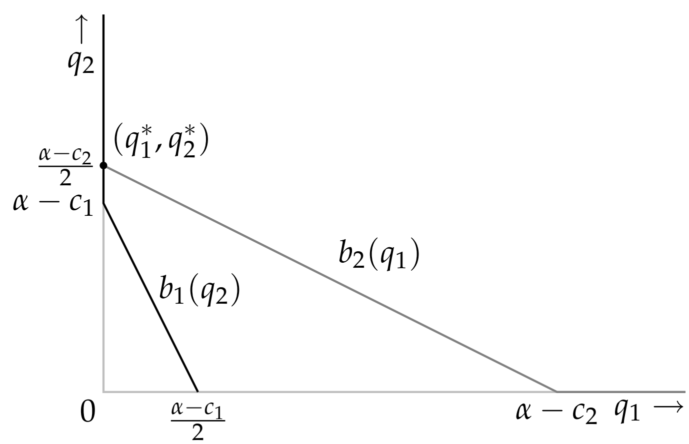

Figure 2. The best response functions in Cournot's duopoly game under the assumptions of Problem 4 when $\alpha - c_1 < \frac{1}{2}(\alpha - c_2)$. The unique Nash equilibrium in this case is $(q_1^*, q_2^*) = (0, \frac{1}{2}(\alpha - c_2))$.