ECO316: Applied game theory
Lecture 2

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2017.9.14

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Example: Investing in a joint project

- n people
- Each person chooses whether to invest
- If at least k people invest, project succeeds (where k is a fixed number with $2 \leq k \leq n - 1$)
- If fewer than k people invest, project fails
- Project succeeds $\Rightarrow$ every investor gets positive return
- Project fails $\Rightarrow$ every investor suffers a loss
- Noninvestors unaffected by outcome of project
- So for every person,

  invest & project succeeds $\succ$ don't invest $\succ$ invest & project fails

Investing in a joint project

Strategic game

- Players: *n* people
- For each player,
  - possible actions: *Invest*, *Don't invest*
  - payoffs: if player chooses *Invest*,

$$\begin{cases} 100 & \text{if at least } k \text{ people choose } \textit{Invest} \\ -10 & \text{if fewer than } k \text{ people choose } \textit{Invest}; \end{cases}$$

  if player chooses *Don't invest*, 0 regardless of others' actions

Investing in a joint project

Nash equilibrium

- Players are symmetric, so we have to think only about whether there is an equilibrium with a certain *number* of investors
- k people invest?
- n people invest?
- no one invests?
- some other number of people invest?

Investing in a joint project

Reminder of payoffs:

- *Invest* $\Rightarrow$ 100 if $\geq k$ investors, -10 if $< k$ investors
- *Don't invest* $\Rightarrow$ 0

Nash equilibrium

- n people invest: Nash equilibrium because player deviates $\Rightarrow$ gets 0 rather than 100
- no one invests: Nash equilibrium because player deviates $\Rightarrow$ gets -10 rather than 0
- between 1 and $k-1$ people invest: not Nash equilibrium because investor deviates $\Rightarrow$ gets 0 rather than -10
- between k and $n-1$ people invest: not Nash equilibrium because noninvestor deviates $\Rightarrow$ gets 100 rather than 0

Investing in a joint project

Reminder of payoffs:

- *Invest* $\Rightarrow$ 100 if $\geq k$ investors, -10 if $< k$ investors
- *Don't invest* $\Rightarrow$ 0

Summary

Exactly two Nash equilibria:

- everyone invests
- no one invests

Traveler's Dilemma

- Airline has lost suitcases of two travelers
- Suitcases and contents are identical
- Airline's process:
  - each traveler specifies value of her suitcase, a number from \$2 to \$100
  - if both travelers specify same number, they are paid that amount
  - if travelers specify different amounts,
    - traveler specifying smaller amount is paid *that amount plus* \$2
    - traveler specifying larger amount is paid *the smaller amount minus* \$2

Traveler's Dilemma

Strategic game

- Players: two travelers
- For each player,
  - possible actions: \$2, \$3, ..., \$100
  - payoffs: for player i,

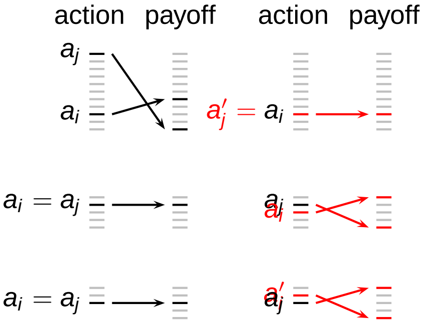
$$\begin{cases} a_i + 2 & \text{if } a_i < a_j \\ a_i & \text{if } a_i = a_j \\ a_j - 2 & \text{if } a_i > a_j \end{cases}$$

where a_i is i's action and a_j is the other player's action

Traveler's Dilemma

Nash equilibrium

- $a_i < a_j$?
  - Not NE: j lowers a_j to a_i $\Rightarrow$ increases j's payoff
- $a_i = a_j$?
  - If $a_i \geq 3$, not NE: i lowers a_i to $a_i - 1 \Rightarrow$ increases i's payoff
  - If $a_i = a_j = 2$, NE! If either player increases amount, payoff $= 0$



Traveler's Dilemma

Summary

Unique Nash equilibrium: both travelers name the lowest possible valuation, \$2

Competition between firms

- Topic at heart of classical economic theory
- How well do privately-owned firms serve consumers' needs?
- Standard model: each firm assumes its own actions have no effect on price
- Each firm takes price as given
- Outcome is independent of number of firms
- Is price-taking assumption reasonable if number of firms is large?
- Need model in which each firm takes others into account
- Can study impact on number of firms on the outcome

Competition between firms

- Firms producing same good compete for customers
- Each firm's profit depends on behavior of all firms
- Model interaction between firms as strategic game
- What are properties of Nash equilibrium?
- How is Nash equilibrium related to "competitive" outcome? How does it depend on number of firms?

Bertrand's model

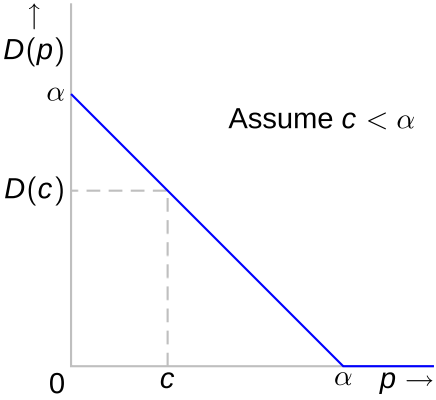
- Each firm chooses a unit price
- Each firm produces enough output to satisfy demand it faces, given prices charged by all firms
- Consumers patronize only firm with lowest price

Strategic game

- Players: firms
- For each firm
  - possible actions: prices
  - payoff: profit

Example of Bertrand's game

- Two firms
- Constant unit cost, same for both firms: $C_i(q_i) = cq_i$ where $c > 0$ and q_i is output of firm i
- Linear demand function: $D(p) = \alpha - p$ for $p \leq \alpha$



Example of Bertrand's game

Cost of firm i's producing q_i units: $C_i(q_i) = cq_i$
Total demand when price is p: $D(p) = \alpha - p$ (for $p \leq \alpha$)

Strategic game

- Players: two firms
- For each firm i,
  - possible actions: prices (nonnegative numbers)
  - payoff: profit, which is

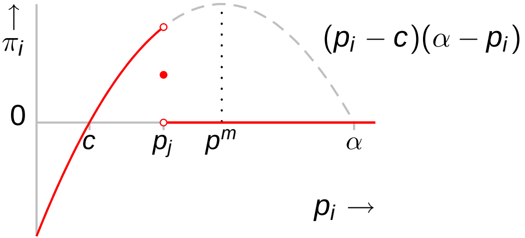
$$\pi_i(p_1, p_2) = \begin{cases} (p_i - c)(\alpha - p_i) & \text{if } p_i < p_j \\ \frac{1}{2}(p_i - c)(\alpha - p_i) & \text{if } p_i = p_j \\ 0 & \text{if } p_i > p_j \end{cases} \quad \text{(assuming } p_i \leq \alpha)$$

where j is the other firm ($j = 2$ if $i = 1$, and $j = 1$ if $i = 2$).

Example of Bertrand's game

Exploration of payoffs:

$$\pi_i(p_1, p_2) = \begin{cases} (p_i - c)(\alpha - p_i) & \text{if } p_i < p_j \\ \frac{1}{2}(p_i - c)(\alpha - p_i) & \text{if } p_i = p_j \\ 0 & \text{if } p_i > p_j \end{cases}$$ (assuming $p_i \leq \alpha$)

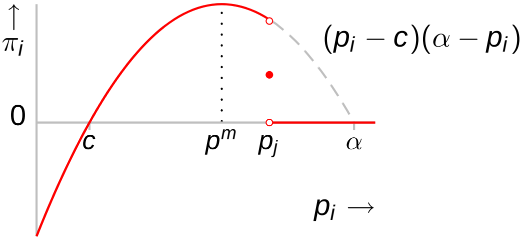


$c < p_j \leq p^m$

Example of Bertrand's game

Exploration of payoffs:

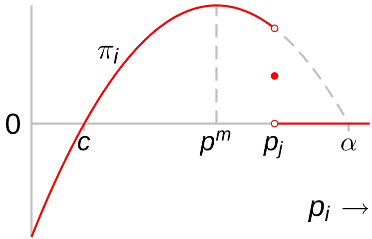
$$\pi_i(p_1, p_2) = \begin{cases} (p_i - c)(\alpha - p_i) & \text{if } p_i < p_j \\ \frac{1}{2}(p_i - c)(\alpha - p_i) & \text{if } p_i = p_j \\ 0 & \text{if } p_i > p_j \end{cases}$$ (assuming $p_i \leq \alpha$)



$$p^m < p_j < \alpha$$

Example of Bertrand's game

Best value of p_i given p_j?



$$p^m < p_j \leq \alpha$$

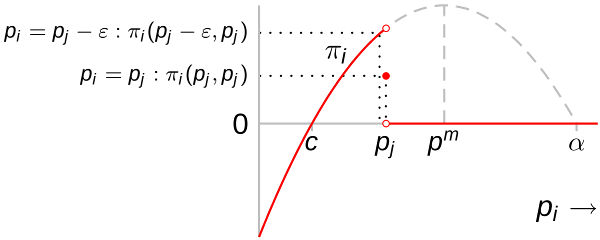
- If $p_j > p^m$, firm i's best price is p^m
- If $p_j < p^m$, firm i can almost double its profit by charging a bit less than p_j

$\Rightarrow$ incentive to "undercut" other firm's price

- Prices less than c yield losses
- So perhaps (c, c) is only equilibrium?

Example of Bertrand's game

Best value of p_i given p_j?



$c < p_j \leq p^m$

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$\Rightarrow$ incentive to "undercut" other firm's price

- Prices less than c yield losses
- So perhaps (c, c) is only equilibrium?

Example of Bertrand's game

Proof that (c, c) is a Nash equilibrium

- $\pi_1(c, c) = 0$
- If $p_1 < c$, then $\pi_1(p_1, c) < 0$
- if $p_1 > c$, then $\pi_1(p_1, c) = 0$

So

$$\pi_1(c, c) \geq \pi_1(p_1, c) \text{ for all } p_1$$

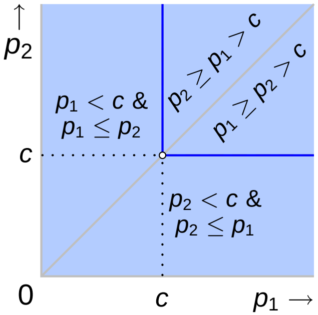
and similarly for firm 2.

Hence (c, c) is a Nash equilibrium.

Example of Bertrand's game

Proof that no pair $(p_1, p_2) \neq (c, c)$ is Nash equilibrium

- $p_1 < c$ and $p_1 \leq p_2$? No: $\pi_1(p_1, p_2) < 0$ and $\pi_1(c, p_2) = 0$, so firm 1 can increase its payoff by deviating to c
- $p_2 < c$ and $p_2 \leq p_1$? No: firm 2 can profitably deviate to c
- $p_1 = c$ and $p_2 > c$? No: firm 1 can profitably *raise* its price: $\pi_1(c, p_2) = 0$ and $\pi_1(p_1, p_2) > 0$ for $c < p_1 < p_2$ and $p_1 < \alpha$
- $p_2 = c$ and $p_1 > c$? No: similar reason
- $p_i \geq p_j > c$? No: firm i can increase its profit by lowering p_i to slightly below p_j if $p_j \leq p^m$ and to p^m if $p_j > p^m$



Bertrand's model: summary

- Unique Nash equilibrium, in which price $=$ unit cost for both firms
- Even with two firms, get the competitive outcome!

Questions

- What about other demand functions?
- What about other cost functions?
- What happens with more than two firms?
- Is there a way for the firms to collude?
- What if firms interact repeatedly?

Cournot's model

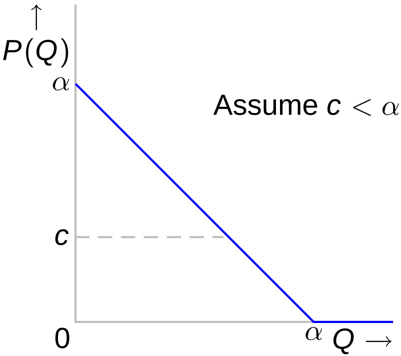
- Each firm chooses an output
- Price is determined by demand function, given firms' total output

Strategic game

- Players: firms
- For each firm
  - possible actions: outputs
  - payoff: profit

Example of Cournot's game

- Two firms
- Constant unit cost, same for both firms: $C_i(q_i) = cq_i$ where $c > 0$ and q_i is output of firm i
- Linear inverse demand function: $P(Q) = \alpha - Q$ for $Q \leq \alpha$



Example of Cournot's game

Cost of firm i's producing q_i units: $C_i(q_i) = cq_i$
Price when supply is Q: $P(Q) = \alpha - Q$ (for $Q \leq \alpha$)

Strategic game

- Players: two firms
- For each firm
  - possible actions: outputs (nonnegative numbers)
  - payoff: profit of firm i

$$\pi_i(q_1, q_2) = \text{revenue} - \text{cost}$$
$$= q_i(\alpha - q_1 - q_2) - cq_i \qquad (\text{if } q_1 + q_2 \leq \alpha)$$

for $i = 1, 2$

Example of Cournot's game

Nash equilibrium

- We want to find a pair (q_1^*, q_2^*) of outputs such that

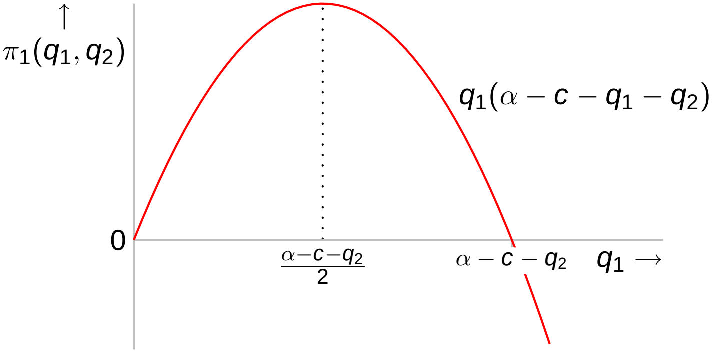
$$q_1^* \text{ is optimal given } q_2^*$$
$$q_2^* \text{ is optimal given } q_1^*$$

- Fix q_2. Which output of firm 1 is optimal given q_2?

$$\text{solution of } \max_{q_1} \pi_1(q_1, q_2)$$
$$\Rightarrow \text{solution of } \max_{q_1} q_1(\alpha - q_1 - q_2) - cq_1$$
$$\Rightarrow \text{solution of } \max_{q_1} q_1(\alpha - c - q_1 - q_2)$$

Example of Cournot's game

Nash equilibrium



Firm 1's payoff as a function of q_1, given q_2

$\Rightarrow$ optimal q_1 given q_2 is $\frac{1}{2}(\alpha - c - q_2)$

Example of Cournot's game

Nash equilibrium

- Can write

$$b_1(q_2) = \frac{1}{2}(\alpha - c - q_2)$$

$\rightarrow$ *best response* of firm 1 to firm 2's output

- Similarly

$$b_2(q_1) = \frac{1}{2}(\alpha - c - q_1)$$

$\rightarrow$ *best response* of firm 2 to firm 1's output

- Nash equilibrium:

$$q_1^* \text{ is optimal given } q_2^*$$
$$q_2^* \text{ is optimal given } q_1^*$$

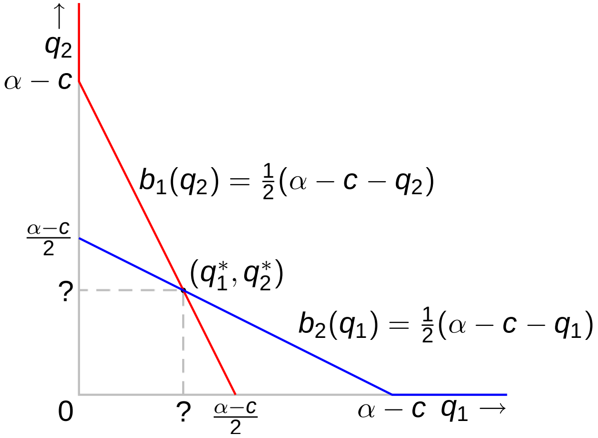
or

$$q_1^* = \frac{1}{2}(\alpha - c - q_2^*)$$
$$q_2^* = \frac{1}{2}(\alpha - c - q_1^*)$$

Example of Cournot's game

Nash equilibrium

$$q_1^* = \frac{1}{2}(\alpha - c - q_2^*) \; (= b_1(q_2^*))$$
$$q_2^* = \frac{1}{2}(\alpha - c - q_1^*) \; (= b_2(q_1^*))$$



Example of Cournot's game

Nash equilibrium

$$q_1^* = \frac{1}{2}(\alpha - c - q_2^*)$$
$$q_2^* = \frac{1}{2}(\alpha - c - q_1^*)$$

$\Rightarrow$

$$q_1^* = \frac{1}{2}(\alpha - c - \frac{1}{2}(\alpha - c - q_1^*))$$
$$q_1^* = \frac{1}{4}(\alpha - c) + \frac{1}{4}q_1^*$$
$$\frac{3}{4}q_1^* = \frac{1}{4}(\alpha - c)$$
$$q_1^* = \frac{1}{3}(\alpha - c)$$

Substitute back to get $q_2^* = \frac{1}{2}(\alpha - c - \frac{1}{3}(\alpha - c)) = \frac{1}{3}(\alpha - c)$

Example of Cournot's game

Nash equilibrium

Conclusion: Unique Nash equilibrium,

$$(q_1^*, q_2^*) = (\tfrac{1}{3}(\alpha - c), \tfrac{1}{3}(\alpha - c))$$

Properties of Nash equilibrium

How is the price in the equilibrium related to unit cost, c?

$$\begin{aligned}\text{Total output } &= q_1^* + q_2^* = \tfrac{1}{3}(\alpha - c) + \tfrac{1}{3}(\alpha - c) = \tfrac{2}{3}(\alpha - c) \\ \Rightarrow \text{price } &= P(\tfrac{2}{3}(\alpha - c)) = \alpha - \tfrac{2}{3}(\alpha - c) = \tfrac{1}{3}(\alpha + 2c)\end{aligned}$$

We have $\alpha > c$, so price $> c$

Example of Cournot's game

Nash equilibrium

Conclusion: Unique Nash equilibrium,

$$(q_1^*, q_2^*) = (\tfrac{1}{3}(\alpha - c), \tfrac{1}{3}(\alpha - c))$$

Properties of Nash equilibrium

How is the equilibrium related to the monopoly outcome?
Monopolist:

$$q^m \text{ solves } \max_q q(\alpha - q) - cq \Rightarrow \max_q q(\alpha - c - q)$$

$$\Rightarrow q^m = \tfrac{1}{2}(\alpha - c)$$

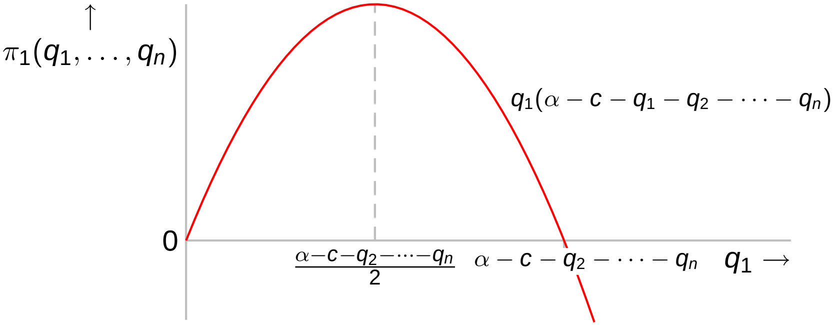
$$\Rightarrow \text{total output in duopoly } > \text{ monopoly output}$$

$$\Rightarrow \text{price in duopoly } < \text{ monopoly price}$$

Example of Cournot's game: many firms

- Suppose number of firms is n, arbitrary number
- Firm 1's payoff function:

$$q_1(\alpha - c - q_1 - q_2 - \cdots - q_n) \quad (\text{if } q_1 + \cdots + q_n \leq \alpha)$$



Firm 1's payoff as a function of q_1, given $q_2, \ldots, q_n$

$\Rightarrow$ optimal q_1 given $q_2, \ldots, q_n$ is $\frac{1}{2}(\alpha - c - q_2 - \cdots - q_n)$

Example of Cournot's game: many firms

- Best response function of firm 1 is

$$b_1(q_{-1}) = \frac{1}{2}(\alpha - c - q_2 - \cdots - q_n) \quad (\text{if } q_2 + \cdots + q_n \leq \alpha - c)$$

  where q_{-1} stands for $(q_2, \ldots, q_n)$
- Other firms' best response functions are same
- $(q_1^*, \ldots, q_n^*)$ is a Nash equilibrium if

$$\begin{aligned} q_1^* &= b_1(q_{-1}^*) \\ q_2^* &= b_2(q_{-2}^*) \\ &\vdots \\ q_n^* &= b_n(q_{-n}^*) \end{aligned}$$

Example of Cournot's game: many firms

- So for Nash equilibrium

$$
\begin{aligned}
q_1^* &= \tfrac{1}{2}(\alpha - c \phantom{{}-q_1^*} - q_2^* - q_3^* - \cdots - q_n^*) \\
q_2^* &= \tfrac{1}{2}(\alpha - c - q_1^* \phantom{{}-q_2^*} - q_3^* - \cdots - q_n^*) \\
&\vdots \\
q_n^* &= \tfrac{1}{2}(\alpha - c - q_1^* - q_2^* - \cdots - q_{n-1}^* \quad)
\end{aligned}
$$

- Multiply each equation by 2:

$$
\begin{aligned}
2q_1^* &= \alpha - c \phantom{{}-q_1^*} - q_2^* - q_3^* - \cdots - q_n^* \\
2q_2^* &= \alpha - c - q_1^* \phantom{{}-q_2^*} - q_3^* - \cdots - q_n^* \\
&\vdots \\
2q_n^* &= \alpha - c - q_1^* - q_2^* - \cdots - q_{n-1}^*
\end{aligned}
$$

Example of Cournot's game: many firms

- From previous slide:

$$\begin{aligned} 2q_1^* &= \alpha - c \qquad\quad - q_2^* - q_3^* - \cdots - q_n^* \\ 2q_2^* &= \alpha - c - q_1^* \qquad\quad - q_3^* - \cdots - q_n^* \\ &\vdots \\ 2q_n^* &= \alpha - c - q_1^* - q_2^* - \cdots - q_{n-1}^* \end{aligned}$$

- Subtract q_i^* from both sides of each equation i:

$$\begin{aligned} q_1^* &= \alpha - c - q_1^* - q_2^* - q_3^* - \cdots - q_n^* \\ q_2^* &= \alpha - c - q_1^* - q_2^* - q_3^* - \cdots - q_n^* \\ &\vdots \\ q_n^* &= \alpha - c - q_1^* - q_2^* - q_3^* - \cdots - q_n^* \end{aligned}$$

- Right-hand side of every equation is the same! So

$$q_1^* = q_2^* = \cdots = q_n^*$$

Example of Cournot's game: many firms

- Let $q_1^* = q_2^* = q_3^* = q_4^* = \cdots = q_n^* = q^*$
- Substitute into

$$q_1^* = \alpha - c - q_1^* - q_2^* - \cdots - q_n^*$$

  (or any of the other equations)
- Result is

$$(n+1)q^* = \alpha - c$$

- So

$$q^* = \frac{\alpha - c}{n+1}$$

Example of Cournot's game: many firms

Conclusion

- Game has unique Nash equilibrium, in which output of every firm i is
$$\frac{\alpha - c}{n+1}$$
- Total output is $n(\alpha - c)/(n+1)$
- Price is $\alpha - n(\alpha - c)/(n+1)$, or
$$\frac{\alpha + nc}{n+1} = \frac{\alpha - c + (n+1)c}{n+1} = c + \frac{\alpha - c}{n+1} > c$$
- As n increases, this price decreases to c
- As number of firms increases, equilibrium outcome approaches competitive outcome

Comparison of Bertrand's and Cournot's games

Bertrand

- strategic variable is price
- firm changes behavior if profit $\uparrow$ assuming other prices don't change (output adjusts)
- Nash equilibrium $\Rightarrow$ price $=$ unit cost (competitive outcome)

Cournot

- strategic variable is output
- firm changes behavior if profit $\uparrow$ assuming other outputs don't change (price adjusts)
- Nash equilibrium $\Rightarrow$ unit cost $<$ price $<$ monopoly price
- Outcome $\rightarrow$ competitive as number of firms increases

Finding Nash equilibrium using best response functions

Technique used to find Nash equilibrium in example of Cournot's game can be used in other games

1. Find the best response function b_i of each player i (optimization problem)
2. Find profiles a^* of actions for which

$$a_1^* = b_1(a_{-1}^*)$$
$$\vdots$$
$$a_n^* = b_n(a_{-n}^*)$$

where a_{-i}^* is the list of actions of the players other than i (typically n equations in n unknowns)

Finding Nash equilibrium using best responses

Example

Players Two people

Actions Each player can choose any nonnegative number

Payoffs Payoff function of player i is $a_i(c + a_j - a_i)$, where $c > 0$ is a constant

1. Find best response function of each player i:

$$b_i(a_j) \text{ solves } \max_{a_i} a_i(c + a_j - a_i) \Rightarrow b_i(a_j) = \tfrac{1}{2}(c + a_j)$$

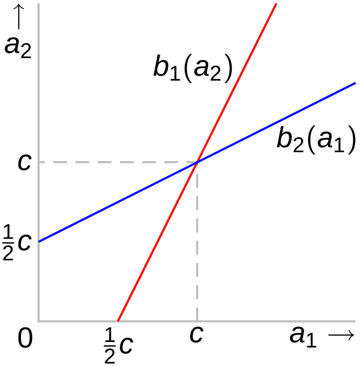
2. Find solution of

$$a_1^* = b_1(a_2^*) = \tfrac{1}{2}(c + a_2^*)$$
$$a_2^* = b_2(a_1^*) = \tfrac{1}{2}(c + a_1^*)$$

Finding Nash equilibrium using best responses

Example

$$a_1^* = b_1(a_2^*) = \tfrac{1}{2}(c + a_2^*)$$
$$a_2^* = b_2(a_1^*) = \tfrac{1}{2}(c + a_1^*)$$



Unique Nash equilibrium: $(a_1^*, a_2^*) = (c, c)$

Finding Nash equilibrium using best responses

Example

Algebraically:

$$\begin{aligned} a_1^* &= \tfrac{1}{2}(c + a_2^*) \\ &= \tfrac{1}{2}(c + \tfrac{1}{2}(c + a_1^*)) \\ &= \tfrac{3}{4}c + \tfrac{1}{4}a_1^* \end{aligned}$$

so

$$\begin{aligned} \tfrac{3}{4}a_1^* &= \tfrac{3}{4}c \\ a_1^* &= c \\ a_2^* &= c \end{aligned}$$

Summary of techniques for finding Nash equilibrium

1. Examine each action profile in turn
   - Used for examples studied last week
2. Intuition, trial and error, and proof
   - Explore game, using intuition to conjecture equilibrium
   - Then prove that action profile selected is equilibrium and no other action profile is equilibrium
   - Used for Bertrand's duopoly game
3. Use best response functions
   - Find best response function b_i of each player i (optimization problem)
   - Find profiles a^* of actions for which

     $$a_i^* = b_i(a_{-i}^*) \text{ for every player } i$$

     where a_{-i}^* is list of actions of other players (typically n equations in n unknowns)

Summary of techniques for finding Nash equilibrium

Best technique depends on game

1. Examine each action profile in turn
   - Possible only if number of action profiles finite!
2. Intuition, trial and error, and proof
   - Useful for games in which economic or other intuition leads to strong conjecture about possible equilibrium
3. Use best response functions
   - Useful for games in which best response functions are easy to compute
   - Can in principle be used for any game, but computing complete best response functions may be difficult and unnecessary (as for Bertrand's game)